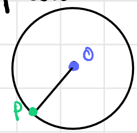
CIRCLES + LINES IN NEUTRAL

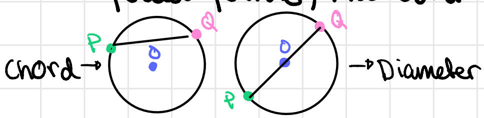
Def 8.1.1: Given a point O and a positive real number r, the circle with centre O and radius r is defined to be the set of all points P st. the distance from O to P is r.



In symbols: $C(O,r) = \{P \mid OP = r\}$

The number r is called the radius of the circle, but often the segment $\overline{OP}$ is a referens to r as well.

Def 8.1.2: Let $C(O,r)$ be a circle. A <u>chord</u> of $C(O,r)$ is a segment joining two points P and Q on $C(O,r)$. Two points P and Q on $C(O,r)$ are said to be antipodal points if $P * O * Q$. In case P and Q are antipodal points, the cord $\overline{PQ}$ is called a diameter of the circle



$\gamma = C(O,r)$

Def 8.1.3: Points A st $OA < r$ are said to be <u>inside</u> the circle $C(O,r)$ while points B st $OB > r$ are <u>outside</u> the sircle

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Thrm 8.1.4: If γ is a circle and ℓ is a line, then the number of points in $\gamma \cap \ell$ is 0, 1 or 2.

Def 8.1.5: A line ℓ is a <u>tangent</u> to circle γ if ℓ intersect γ in exactly one point. If P is the point at witch ℓ intersect γ, then we say that ℓ is tangent to γ at P. A segment $\overline{AB}$ is said to be tangent to a circle γ if $\overline{AB}$ is containd in a line that is tangend to γ and the point of tangency is an interior point of $\overline{AB}$.

Def 8.1.6: A line ℓ is a <u>sacant line</u> for circle γ if ℓ intersects γ in two distinct points.

Thrm 8.1.7 (Tangent Line Thrm): Let t be a line, $\gamma = C(O,r)$ a circle, and P a point of $t \cap \gamma$. The line t is a tangent to the circle at the point P if and only if $\overleftrightarrow{OP} \perp t$.

Thrm 8.1.8: If γ is a circle and t is a tangent line, then every point of t except for P is outside γ.

Thrm 8.1.9 (Secant Line Thrm): If $\gamma = C(O,r)$ is a circle and l is a secant line that intersects γ at distinct points P and Q, then O lies on the perpendicular bisector of the chord $\overline{PQ}$

Thrm 8.1.10: If γ is a circle and l is a secant line that l intersects γ at points P and Q, then every point on the interior of $\overline{PQ}$ is inside γ and everypoint of $l \setminus \overline{PQ}$ is outside PQ.

Thrm 8.1.11 (Elementary Circular Continuity): If γ is a circle and l is a line st l contains a point A that is inside γ and a Point B that is outside γ, then l is a secant line for γ

Corollary 8.1.12: If γ is a circle and l is a line st l contains a point A that is inside γ, then l is a secant line for γ.

Def 8.1.14: Two circles $\gamma_1 = C(O_1, r_1)$ and $\gamma_2 = C(O_2, r_2)$ are tangents if $\gamma_1 \cap \gamma_2$ consists of exactly one point.

Thrm 8.1.15 (Tangent Circles Thrm): If the circle $\gamma_1 = C(O_1, r_1)$ and $\gamma_2 = (O_2, r_2)$ are tangents at P, then the centers O_1 and O_2 are distigt and three points O_1, O_2 and P are collinear.
Futhermore, the circles share a common tangent line at P.

CIRCLES + TRIANGLES IN NEUTRAL GEOMETRY

Def 8.2.1: A circle that contains all three vertices of the triangle $\triangle ABC$ is said to circumscribe the triangle. The circle is called the circumscircle and the center is called the circumcenter of the triangle. If a triangle has a circumcircle, we say the triangle can be circumbscribed

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Thrm 8.2.2 (Circumscribed Circle Thrm): A triangle can be circumscribed if and only if the perpendicular bisectors of the sides of the triangle are concurrent. If a triangle can be circumscribed, then the circumcenter and the circumcircle are unique.

Thrm 8.2.3: The Euclidian Parallel Post, is equivelant to the assertion that every triangle can be circumscribed

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Thrm 8.2.4: If the Euclidean Parallel Post. holds, then every triangle can be circumscribed.

Corollary 8.2.5: In Euclidean geo. the three perpendicular bisector of the sides of any triangle are congurrent and meet at the circumcenter of the triangle

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Thrm 8.2.6: If the Euclidean Parallel Post. fails then there exists a triangle that cannot be circumscribed (Hyperbolic)

Def 8.2.7: Let $\triangle ABC$ be a triangle. A circle $C(O,r)$ is called the inscribed circle per $\triangle ABC$ if each of the segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ is tangent to $C(O,r)$. The centre of the inscribed circle is called the intercentre of the triangle

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Thrm 8.2.8 (Inscribed Cirle Thrm): Every triangle has a unique insuribed circle. The bisectors of the interior angles in any triangle are concurrent ant the point of concurrency is the incentre of the triangle.

Def 8.2.9: Fix an integer $n \geq 3$. Suppose $P_1, P_2, \ldots, P_n$ are n distigt points st no three points are collinear. Suppose further that any two segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}$ and $\overline{P_1P_n}$ are either disjoint or share an endpoint. If those conditions are satisfied, the points $P_1, P_2, \ldots, P_n$ determine a polygon. The Polygon is defined to be a union of the segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots \overline{P_{n-1}P_n}$ and is denoted by $P_1P_2 \cdots P_n$

In symbols: $P_1P_2 \cdots P_n = \overline{P_1P_2} \cup \overline{P_2P_3} \cup \cdots \cup \overline{P_{n-1}P_n} \cup \overline{P_nP_1}$

Def 8.2.10: A polygon $P_1P_2 \cdots P_n$ is a regular polygon if all the segments are congurent to each other and all the angles are congurent to each other.

Def 8.2.11: A polygon $P_1P_2 \cdots P_n$ is said to be inscribed in the circle γ if all the vertices of $P_1P_2 \cdots P_n$ lie on γ.

Thrm 8.2.12: Let γ be a circle and P_1 a point on γ. For each $n \geq 3$ there is a regular polygon $P_1P_2 \cdots P_n$ inscribed in γ.

CIRCLES IN EUCLIDEAN GEOMETRY

Thrm 8.3.1: Let ΔABC be a triangle and let M be the midpoint of $\overline{AB}$. If $AM = MC$, then $\measuredangle ACB$ is a right angle.

Corollary 8.3.2: If the vertices of triangle ΔABC lie on a circle and $\overline{AB}$ is a diameter of that circle, then $\measuredangle ACB$ is a right angle.

Thrm 8.3.3: Let ΔABC be a triangle and let M be the midpoint of $\overline{AB}$. If $\measuredangle ABC$ is a right angle, then $AM = MC$.

Corollary 8.3.4: If $\measuredangle ABC$ is a right angle, then $\overline{AB}$ is a diameter of the circle that circumscribes ΔABC

Thrm 8.3.5 (The 30-60-90 Thrm): If the interior angles in triangle ΔABC measure $30°$, $60°$ and $90°$, then the length of the side opposite the $30°$ angle is one half of the length of the hypotenuse.

Thrm 8.3.6 (Converse to the 30-60-90) If ΔABC is a right triangle st the length of one leg is one-half the length of the hypotenuse, then the interior angles of the triangles measure $30°$, $60°$, and $90°$

Thrm 8.3.9 (Central Angle Thrm): The measure of an inscribed angle for a circle is one half the measure of the corresponding central angle.

Corollary 8.3.10 (Inscribed Angle Thrm): If two inscribed angles intercept the same arc, then the angles are congruent.

Thrm 8.3.12: The power of a points is well defined; that is, the same value is obtained regardless of witch line ℓ is used in the definition as long as the line has one point of intersection with the circle.

THE CARTESIAN MODEL

Point: an ordered pair of real numbers (x,y)

Line: a particular type of set of points.
↳ For each triple of real numbers a, b, c with a and b not both 0 there is a line ℓ defined $\ell = \{(x,y) \mid ax+by+c=0\}$

Distance: Defined by the usual formula. Let $A=(x_1, y_1)$ and let $B=(x_2, y_2)$. The distance from A to B:
$$AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

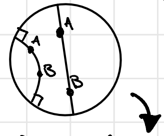
Half-plane: if ℓ is a line defined by: $ax+by+c=0$. Half-plane determined by ℓ: $H_1 = \{(x,y) \mid ax+by+c>0\}$
and $H_2 = \{(x,y) \mid ax+by+c<0\}$

Angle measure: the inverse tangent function:
$$\arctan x = \int_0^x \frac{1}{1+t^2} \cdot dt$$
Equation for two nonvertical lines; write in the form
$y=m_1x+b_1$ and $y=m_2x+b_2$.
If $m_1m_2=-1$, then the lines are perpendicular.
Otherwise, the smaller angle between them has measure
$$\left(\frac{180}{\pi}\right)\left|\arctan\left(\frac{m_1-m_2}{1+m_1m_2}\right)\right|$$
The measure of the larger angle between the two lines are obtained by subtracting the value above from 180°.

POINCARÉ DISK MODEL

Fix a circle γ in the Euclidean plane.
↳ In Cartesian model take γ to be the circle of radius 1 centered at the origen.

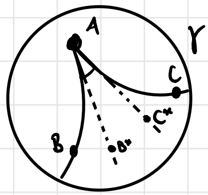
Point: Is a Euclidean point inside γ



Line: Two ways of interpretation:
- A line is a open diameter of γ. Starts with a Euclidean line l that passes through the centre of γ. The assosiated Poincaré line consists of all the points on l that are inside γ
- A line that starts with a Euclidean circle β that is orthogonal to γ. The assosiated Poincaré line consists of all the points on β that lie inside γ

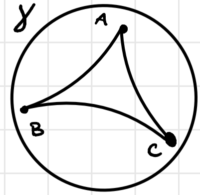
Half-Plane: let m be a line in the model.
- If m is a line of the first kind then it is determined by a Euclidean line l that passes trough the centre of γ. There are two halfplanes determined by l. The Poincaré half-planes determined by m are defined to be the intersections of the Euclidean half-planes with the interior of γ.
- In case m is determined by the Euclidean circle β, we define the halfplane determined by m to be the intersection of the interior and exterion of β with the interior of γ.

Angle measure: Points on either type of Poincaré line have a natural notion of betweenness that is inherited from Euclidean geometry.
We will assume this definition of betweenness and rays are defined in Poincaré disk model. Two Poincaré rays determine two Euclidean tangent rays and we define the measure of the angle between the Poincaré rays to be the measure of the Euclidean angle between the tangent rays.



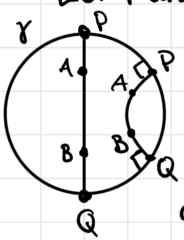
Conformal model:

-Two models are conformal



Triangles: The angles have smaller measure than the angles would have in a Euclidean triangle, the triangle has positive defect.

Distance: Let A and B be distict points in the model. If A and B lie on the diameter of γ, then define P and Q be the endpoints of the diameter. If A and B do not lie on a diameter, then there is a unique Euclidean circle β that contains the two points and is perpendicular to γ. Let P and Q be two points at witch β intersects γ



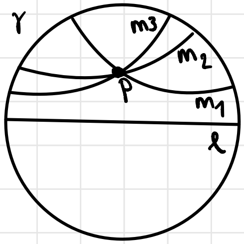
P and Q are points on γ and therefore not points in the Poincaré disk model of hyperbolic geometry. Points on γ are called <u>ideal points</u> for the model and can be thoght of as being infinilly far from any ordinary points of the model.

Def 11.2.1: Define distance from A to B by $d(A,B) = |\ln([AB,PQ])|$, where $[AB,PQ]$ is the cross ratio.

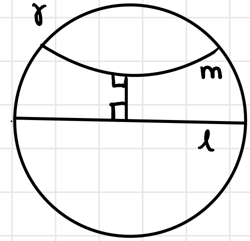
$[AB,PQ]$ is defined by: $[AB,PQ] = \dfrac{(AP)(BQ)}{(AQ)(BP)}$, and also

$[BA,PQ] = \dfrac{1}{[AB,PQ]} = [AB,QP]$. It follows that d is symetric in the sense that $d(B,A) = |\ln([BA,PQ])| = |\ln([AB,PQ])| = d(A,B)$

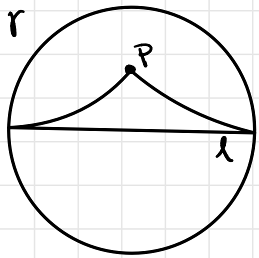
Various Parallel lines in Poincaré disk model:



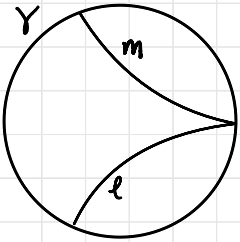
multiple parallel lines through a point.



a common prependicular segment.



two limiting parallel rays from P



Asymptotically parallel lines