Formulas Collection MA2401 - Geometry

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T = Theorem, C = Corollary, L = Lemma, A = Axiom, definitions are omitted.

Axiomatic Systems and Geometry Incidence

IA 1: For every pair of distinct points P and Q there exists exactly one line l such that both P and Q lie on l.

IA 2: For every line l there exists at least two distinct points P and Q such that both P and Q lie on l.

IA 3: There exists three points that do not all lie on any one line.

Euclidian Parallel Postulate: For every line l and for every point P that does not lie on l, there is exactly one line m such that P lies on m and $m \parallel l$.

Elliptic Parallel Postulate: For every line l and for every point P that does not lie on l, there is no line m such that P lies on $m \parallel l$.

Hyperbolic Parallel Postulate: For every line l and for every point P that does not lie on l, there are at least two lines m and n such that P lies on both m and n, and both m and n are both parallel to l.

T 2.6.3: If l is any line, there exists at least one point P such that P does not lie on l.

T 2.6.4: If P is any point, then there are at least two distinct lines l and m such that P lies on both l and m.

T 2.6.5: If l is any line, then there exists lines m and n such that l,m and n are distinct and both m and n intersect l.

T 2.6.6: If P is any point, then there exists at least one line l such that P does not lie on l.

T 2.6.7: There exist three distinct lines such that no point lies on all three of the lines.

T 2.6.8: If P is any point, then there exist points Q and R such that P,Q and R are noncollinear.

T 2.6.9: If P and Q are two points such that $P \neq Q$, then there exists a point R such that P,Q and R are noncollinear.

Axioms for Plane Geometry

A 3.1.1 (Existence Postulate): The collection of all points forms a nonempty set. There is more than one point in that set.

A 3.1.3 (Incidence Postulate): Every line is a set of points. For every pair of distinct points A and B there is exactly one line l such that $A \in l$ and $B \in l$.

T 3.1.7: If l and m are two distinct, nonparallel lines, then there exists exactly one point P such that P lies on both l and m.

A 3.2.1 (Ruler Postulate): For every pair of points P and Q there exists a real number PQ, called the distance from P to Q. For each line l there is a one-to-one correspondence from l to $\mathbb{R}$ such that if P and Q are points on that line that corresponds to the real numbers x and y, then $PQ = |x - y|$.

T 3.2.7: If P and Q are any two points, then:
1. $PQ = QP$,
2. $PQ \geq 0$, and
3. $PQ = 0$ if and only if $P = Q$

C 3.2.8: $A * C * B$ if and only $B * C * A$.

3.2.16 (Ruler Placement Postulate): For every pair of distinct points P and Q, there is a coordinate function $f : \overleftrightarrow{PQ} \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

T 3.2.17 (Betweenness for Points): Let l be a line; A, B, and C be three distinct points that all lie on l; and $f : l \to \mathbb{R}$ be a coordinate function for l. The point C is between A and B if and only if either $f(A) < f(C) < f(B)$ or $f(A) > f(C) > f(B)$.

C 3.2.18: Let A, B, and C be three points such that B lies on $\overrightarrow{AB}$. Then $A * B * C$ if and only if $AB < AC$.

C 3.2.19: If A, B, and C are three distinct collinear points, then exactly one of them lies between the other two.

C 3.2.20: Let A and B be two distinct points. If f is a coordinate function for $l = \overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$, then $\overrightarrow{AB} = \{P \in l \mid f(P) \geq 0\}$.

T 3.2.22 (Existence, Uniqueness for Midpoints): If A and B are distinct points, then there exists a unique point M such that M is the midpoint of $\overline{AB}$.

T 3.2.23 (Point Construction): If A and B are distinct points and d is any nonnegative real number, then there exists a unique points C such that C lies on $\overrightarrow{AB}$ and $AC = d$.

A 3.3.2 (Plane Separation): For every line l, the points that do not lie on l form two disjoint, nonempty sets H_1 and H_2, called half-planes bounded by l, such that the following conditions are satisfied:
1. Each of H_1 and H_2 is convex.
2. If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ}$ intersects l.

T 3.3.9 (Ray Theorem): Let l be a line, A a point on l, and B an external point for l. If C is a point on $\overrightarrow{AB}$ and $C \neq A$, then B and C are on the same side of l.

T 3.3.10: Let A, B, and C be three noncollinear points and let D be a point on the line $\overleftrightarrow{BC}$. The point D is between points B and C if and only if the ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

T 3.3.12 (Pasch's Axiom): Let $\triangle ABC$ be a triangle and let l be a line such that none of A, B, and C lies on l. If l intersects $\overline{AB}$, then l also intersects either $\overline{AC}$ or $\overline{BC}$.

A 3.4.1 (Protractor Postulate): For every angle $\angle BAC$ there is a real number $\mu(\angle BAC)$, called the measure of $\angle BAC$, such that the following conditions are satisfied:
1. $0° \leq \mu(\angle BAC) < 180°$ for every angle $\angle BAC$
2. $\mu(\angle BAC) = 0°$ if and only if $\overrightarrow{AB} = \overrightarrow{AC}$
3. For each real number r, $0 < r < 180$, and for each half-plane H bounded by $\overleftrightarrow{AB}$ there exists a unique ray $\overrightarrow{AE}$ such that E is in H and $\mu(\angle BAE) = r°$
4. If the ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)$.

L 3.4.4: If A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overleftrightarrow{AC}$, then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.

T 3.4.5 (Betweenness Rays): Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $\mu(\angle BAD) < \mu(\angle BAC)$ if and only if $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

T 3.4.7 (Existence and Uniqueness for Angle Bisectors): If A, B, and C are three noncollinear points, then there exists a unique angle bisector for $\angle BAC$.

T 3.5.1 (Z-Theorem): Let l be a line and let A and D be distinct points on l. If B and E are points on opposite sides of l, then $\overrightarrow{AB} \cap \overrightarrow{DE} = \emptyset$.

T 3.5.2 (Crossbar): If $\triangle ABC$ is a triangle and D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

T 3.5.3: A point D is in the interior of the angle $\angle BAC$ if and only if the ray $\overrightarrow{AD}$ intersects the interior of the segment $\overline{BC}$.

T 3.5.5 (Linear Pair): If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180°$.

L 3.5.7: If $C * A * B$ and D is in the interior of $\angle BAE$, then E is in the interior of $\angle DAC$.

T 3.5.9: If l is a line and P is a point on l, then there exists exactly one line m such that P lies on m and $m \perp l$.

T 3.5.11 (Existence, Uniqueness Perpendicular Bisectors): If D and E are two distinct points, then there exists a unique perpendicular bisector of $\overline{DE}$.

T 3.5.13 (Vertical Angles): Vertical angles are congruent.

L 3.5.14: Let $[a, b]$ and $[c, d]$ be closed intervals of real numbers and let $f : [a, b] \to [c, d]$ be a function. If f is strictly increasing and onto, then f is continuous.

A 3.6.3 (SAS): If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

T 3.6.5 (Isosceles): The base angles of an isosceles triangle are congruent.

Neutral Geometry

T 4.1.2 (YVT): The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.

T 4.1.3 (Existence, Uniqueness Perpendiculars): For every line l and for every point P, there exists a unique line m such that P lies on m and $m \perp l$.

T 4.2.1 (ASA): If two angles and the included side of one triangle are congruent to the corresponding parts of a second triangle, then the two triangles are congruent.

T 4.2.2 (Converse Isosceles): If $\triangle ABC$ is a triangle such that $\angle ABC \cong \angle ACB$, then $\overline{AB} \cong \overline{AC}$.

T 4.2.3 (AAS): If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.

T 4.2.5 (Hypotenuse-Leg): If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the two triangles are congruent.

T 4.2.6: If $\triangle ABC$ is a triangle, $\overline{DE}$ is a segment such that $\overline{DE} \cong \overline{AB}$, and H is a hal-plane bounded by $\overleftrightarrow{DE}$, then there is a unique point $F \in H$ such that $\triangle DEF \cong \triangle ABC$.

T 4.2.7 (SSS): If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

T 4.3.1 (Scalene): Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.

T 4.3.2 (Triangle Inequality): If A, B, and C are three noncollinear points, then $AC < AB + AC$.

T 4.3.3 (Hinge): If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $AB = DE$, $AC = DF$, and $\mu(\angle BAC) < \mu(\angle EDF)$, then $BC < EF$.

T 4.3.4: Let l be a line, let P be an external point, and let F be the foot of the perpendicular from P to l. If R is any point on l that is different from F, then $PR > PF$.

T 4.3.6 (Pointwise Characterization Angle Bisector): Let A, B, and C be three noncollinear points and let P be a point in the interior og $\angle BAC$. Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$.

T 4.3.7 (Pointwise Characterization Perpendicular Bisector): Let A and B be distinct points. A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$.

T 4.4.2 (AIVT): If l and l' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then $l \perp l'$.

C 4.4.4 (Corresponding Angles): If l and l' are lines cut by a transversal t in such a way that two corresponding angles are congruent, then l is parallel to l'.

C 4.4.5: If l and l' are lines cut by a transversal t in such a way that two nonalternating interior angles on the same side of t are supplements, then l is parallel to l'.

C 4.4.6 (Existence Parallels): If l is a line and P is an external point, then there is a line m such that P lies on m and m is parallel to l.

C 4.4.7: The Elliptic Parallel Postulate is false in any model for neutral geometry.

C 4.4.8: If l, m, and n are three lines such that $m \perp l$ and $n \perp l$, then either $m = n$ or $m \parallel n$.

T 4.5.2 (Saccheri-Legendre): If $\triangle ABC$ is any triangle, then $\sigma(\triangle ABC) \leq 180°$.

L 4.5.3: If $\triangle ABC$ is any triangle,
$\mu(\angle CAB) + \mu(\angle ABC) < 180°$.

L 4.5.4: If $\triangle ABC$ is a triangle and E is a point in the interior og $\overline{BC}$, then
$\sigma(\triangle ABE) + \sigma(\triangle ECA) = \sigma(\triangle ABC) + 180°$.

L 4.5.5: If A, B, and C are three noncollinear points, then there exists a point D that does not lie on $\overleftrightarrow{AB}$ such that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$ and the angle measure of one of the interior angles in $\triangle ABD$ is less than or equal to
$\frac{1}{2}\ \mu(\angle CAB)$

C 4.5.6: The sum of the measures of two interior angles of a triangle is less than or equal to the measure of their remote exterior angle.

C 4.5.7 (Converse Euclid's fifth): Let l and l' be two lines cut by a transversal t. If l and l' meet on one side of t, then the sum of the measures of the two interior angles on that side of t is strictly less than 180°.

T 4.6.4: If $\square ABCD$ is a convex quadrilateral, then
$\sigma(\square ABCD) \leq 360°$.

T 4.6.6: Every parallelogram is a convex quadrilateral.

T 4.6.7: If $\triangle ABC$ is a triangle, D is between A and B, and E is between A and C, then $\square BCED$ is a convex quadrilateral.

T 4.6.8: The quadrilateral $\square ABCD$ is convex if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

C 4.6.9: If $\square ABCD$ and $\square ACBD$ are both quadrilaterals, then $\square ABCD$ is not convex. If $\square ACBD$ is a quadrilateral.

T 4.7.3+ Each of the following statements is equivalent to the Euclidian Parallel Postulate: **1.** Proclus's Axiom: If l and l' are parallel lines and $t \neq l$ is a line such that t intersects l, then t' also intersects l'.
2. If l and l' are parallel lines and t is a transversal such that $t \perp l$, then $t \perp l'$
3. If l, m, n, and k are lines such that $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.
4. Transistivity of Parallelism: If l is parallel to m and m is parallel to n, then either $l = n$ or $l \parallel n$.
5. Converse AIVT: If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.
6. Euclid's Postulate V: If l and l' are two lines but by a transversal t in such a way that the sum of the measures of the two interior angles on one side of t is less than $180°$, then l and l' intersects on that side of t.
7. Hilbert's PP: For every line l and for every external point P there exists at most one line m such that P lies on m and $m \parallel l$.
8. Angle Sum: If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180°$.
9. Wallis's Postulate: If $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment, then there exists a point F such that $\triangle ABC \sim \triangle DEF$.
10. Clairaut's Axiom: There exists a rectangle.

L 4.7.5 Suppose $\overline{PQ}$ is a segment and Q' is a point such that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there exists a point T on $\overrightarrow{QQ'}$ such that $\mu(\angle PTQ) < \epsilon°$.

T 4.8.2 (Additivity Defect):
1. If $\triangle ABC$ is a triangle and E is a point in the interior of $\overline{BC}$, then:
$\delta(\triangle ABC) = \delta(\triangle ABE) + \delta(\triangle ECA)$.
2. If $\square ABCD$ is a convex quadrilateral, then
$\delta(\square ABCD) = \delta(\triangle ABC) + \delta(\triangle ACD)$.

T 4.8.4: The following statements are equivalent:
1. There exists a triangle whose defect is $0°$.
2. There exists a right triangle whose defect is $0°$.
3. There exists a rectangle.
4. There exist arbitrarily LARGE rectangles.
5. The defect of every right triangle is $00°$.
6. The defect of every triangle is $0°$.

C 4.8.5: In any model for neutral geometri, there exists one triangle qhose defect is $o°$ if and only if every triangle has defect $o°$.

L 4.8.6: If $\triangle ABC$ is any triangle, then at least two of the interior angles in $\triangle ABC$ are acute. If the interior angles at vertices A and B are acute, then the foot of the perpendicular from C to $\overleftrightarrow{AB}$ is between A and B.

T 4.8.10 (Saccheri quadrilaterals): If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then:
1. The diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.
2. The summit angles $\angle BCD$ and $\angle ADC$ are congruent.
3. The segment joining the midpoint of $\overline{AB}$ to the midpoint of $\overline{CD}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$.
4. $\square ABCD$ is a parallelogram.
5. $\square ABCD$ is a convex quadrilateral.
6. The summit angles $\angle BCD$ and $\angle ADC$ are either right or acute.

T 4.8.11 (Lambert quadrilaterals): If $\square ABCD$ is a Lambert quadrilateral with right angles at vertices A, B, and C, then:
1. $\square ABCD$ is a parallelogram.
2. $\square ABCD$ is a convex quadrilateral.
3. $\angle ADC$ is either right or acute.
4. $BC \leq AD$.

T 4.8.12 (Aristotle): If A, B, and C are three noncollinear points such that $\angle BAC$ is an aute angle and P and Q are two points on $\overrightarrow{AB}$ with $A * P * Q$, then $d(P, \overleftrightarrow{AC}) < d(d, \overleftrightarrow{AC})$. Furthermore, for every positive number d_0 there exists a point R on $\overrightarrow{AB}$ such that $d(R, \overleftrightarrow{AC}) > d_0$.

T 4.9.1 (Universal Hyperbolic): If there exists one line l_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to l_0, then for every line l and for every external point P there exist at least two lines that pass through P and are parallel to l.

Euclidian Geometry

T 5.1.1 (Converse AIVT): If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.

T 5.1.2 (Euclid V): If l and l' are two lines cut by a transversal t such that the sum of the measures of the two interior angles on side of t is less than $180°$, then l and l' intersect on that side of t.

T 5.1.3 (Angle Sum): For every triangle $\triangle ABC$, $\sigma(\triangle ABC) = 180°$.

T 5.1.4 (Wallis's Postulate): If $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment, then there exists a point F such that $\triangle ABC \sim \triangle DEF$.

T 5.1.5 (Proclus's Theorem): If l and l' are parallel lines and $t \neq l$ is a line such that t intersects l, then t also intersects l'.

T 5.1.6: If l and l' are parallel lines and t is a transversal such that $t \perp l$, then $t \perp l'$.

T 5.1.7: If l, m, n, and k are lines such that $k \| l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \| n$.

T 5.1.8 (Transistivity Parallelism): If $l \| m$ and $m \| n$, then either $l = n$ or $l \| n$.

T 5.1.9 (Claiaut's Axiom): There exists a rectangle.

T 5.1.10 (Euclidian Rectangles): If $\square ABCD$ is a parallelogram, then:
1. The diagonals divide the quadrilateral into congruent triangles (i.e., $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$).
2. The opposite sides are congruent (i.e., $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$).
3. The opposite angles are congruent (i.e., $\angle DAB \cong \angle BCD$ and $\angle ABC \cong \angle CDA$).
4. The diagonals bisect each other (i.e., $\overline{AC}$ and $\overline{BD}$ intersect in a point E that is the midpoint of each.

T 5.2.1 (Parallel Projection): Let l, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a tranversal that cuts the lines at points A', B', and C', respectively. Assume that $A * B\ astC$. Then $\frac{AB}{AC} = \frac{A'B'}{A'C'}$.

L 5.2.2: Let l, m, and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts the lines at points A', B', and C', respecitvely. Assume $A * B * C$. If $\overline{AB} \cong \overline{BC}$, then $\overline{A'B'} \cong \overline{B'C'}$.

T 5.3.1 (Fundamental Similar Triangles): IF $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then $\frac{AB}{AC} = \frac{DE}{DF}$.

C 5.3.2: If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then there is a positive number r sych that $DE = r \cdot AB$, $DF = r \cdot AC$, and $EF = r \cdot BC$.

T 5.3.3 (SAS Similarity Criterion): If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$ and $\frac{AB}{AC} = \frac{DE}{DF}$, then $\triangle ABC \sim \triangle DEF$.

T 5.4.1 (Pythagoras): If $\triangle ABC$ is a right triangle with right angle at vertex C, then $a^2 + b^2 = c^2$.

T 5.4.3: The height of a right triangle is the geometric mean of the lenghts of the projections of the legs.

T 5.4.4: The length of one leg of a right triangle is the geometric mean of the length of the hypotenuse and the length of the projection of that leg onto the hypotenuse.

T 5.4.5 (Converse Pythagoras): If $\triangle ABC$ is a triangle such that $a^2 + b^2 = c^2$, then $\angle BCA$ is a right angle.

T 5.5.2 (Pythagorean Identity): For any angle θ, $sin^2\theta + cos^2\theta = 1$.

T 5.5.3 (Law of sines): If $\triangle ABC$ is any triangle, then $\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$.

T 5.5.4 (Law of cosines): If $\triangle ABC$ is any triangle, then $c^2 = a^2 + b^2 - 2ab\ \cos C$.

T 5.6.2 (Median Concurrence):The three medians of any triangle are concurrent; that is, if $\triangle ABC$ is a triangle and D, E, and F are the midpoints of the sides opposite A, B, and C, respectively, then $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ all intersect in a common point G. Moreover, $AG = 2GD$, $BG = 2GE$, and $CE = 2GF$.

T 5.6.3 (Euler Line): The orthocenter H, the circumcenter O, and the centroid G of any triangle are collinear. Furthermore, G is between H and O (unless the triangle is equilateral, in which case the three points coincide) and $HG = 2GO$.

T 5.6.4 (Ceva's): Let $\triangle ABC$ be a triangle. The proper Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent (or mutually parallel) if and only if:
$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$.

T 5.6.5 (Menelaus): Let $\triangle ABC$ be a triangle. Three proper Menelaus points L, M, and N on the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$, respectively, are collinear if and only if:
$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$.

T 5.6.6: If $\triangle$ is any triangle, then the assiciated Morley triangle is equilateral.

Hyperbolic Geometry

T 6.1.1: For every triangle $\triangle ABC$, $\sigma(\triangle ABC) < 180°$.

C 6.1.2: For every triangle $\triangle ABC$,
$0° < \delta(\triangle ABC) < 180°$

T 6.1.3: For every convex quadrilateral $\square ABCD$, $\sigma(\square ABCD) < 360°$.

C 6.1.4: The summit angles in a Saccheri quadrilateral are acute.

C 6.1.5: The fourth angle in a Lambert quadrilateral is acute.

T 6.1.6: There does not exist a rectangle.

T 6.1.7: In a Lambert quadrilateral, the length of a side between two right angles is strictly less than the length of the opposite side.

C 6.1.9: In a Saccheri quadrilateral, the length of the altitude is less than the length of a side.

C 6.1.10: In a Saccheri quadrilateral, the length of the summit is greater than the length of the base.

T 6.1.11 (AAA): If $\triangle ABC$ is similar to $\triangle DEF$, then $\triangle ABC$ is congruent to $\triangle DEF$.

T 6.1.12: If $\square ABCD$ and $\square A'B'C'D'$ are two Saccheri quadrilaterals such that $\delta(\square ABCD) = \delta(\square A'B'C'D')$ and $\overline{CD} \cong \overline{C'D'}$, then $\square ABCD \cong \square A'B'C'D'$.

T 6.2.1: If l is a line, P is an external point, and m is a line such that P lies on m, then there exists at most one point Q such that $Q \neq P$, Q lies on m, and $d(Q, l) = d(Q, l)$.

T 6.2.3: If l and m are parallel lines and there exist two points on m that are equidistant from l, then l and m admit a common perpendicular.

T 6.2.4: If lines l and m admit a common perpendicular, then that common perpendicular is unique.

T 6.2.4: Let l and m be parallel lines cut by a transversal t. Alternate interior angles formed by l and m with transversal t are congruent if and only if l and m admit a common perpendicular and t passes through the midpoint of the common perpendicular segment.

T 6.3.2: Let K be the intersecting set for P and $\overrightarrow{AB}$. If $r \in K$, then $s \in K$ for every s with $0 < s < r$, and there exists $t \in K$ such that t > r.

T 6.3.5: The critical number depends only on $d(P, l)$.

T 6.3.7: $\kappa : (0, \infty) \to (0, 90]$ is a nonincreasing function; that is, $a < b$ implies $\kappa(a) \geq \kappa(b)$.

T 6.3.8: Every angle of parallelism is acute and every critical number is less than 90.

Circles

T 8.1.4: If γ is a circle and l is a line, then the number of points in $\gamma \cap l$ is 0, 1 or 2.

T 8.1.7 (Tangent Line): Let t be a line, $\gamma = \mathcal{C}(O, r)$ a circle, and P a point of $t \cap \gamma$. The line t is tangent to the circle γ at the point P if and only if $\overleftrightarrow{OP} \perp t$.

T 8.1.8: If γ is a circle and t is a tangent line, then every point of t execpt for P is outside γ.

T 8.1.9 (Secant Line): If $\gamma = \mathcal{C}(O, r)$ is a circle and l is a secant line that intersects γ at distinct points P and Q, then O lies on the perpendicular bisector of the chord $\overline{PQ}$.

T 8.1.10: If γ is a circle and l is a secant line such that l intersects γ at points P and Q, then every point on the interior of $\overline{PQ}$ is inside γ and every point of $l \setminus \overline{PQ}$ is outside γ.

T 8.1.11 (Elementary Circular Continuity): If γ is a circle and l is a line such that l contains a point A that is inside γ and a point B that is outside γ, then l is a secant line for γ.

C 8.1.12: If γ is a circle and l is a line such that l contains a point A that is inside γ, then l is a secant line for γ.

T 8.1.15 (Tangent Circles): If the circles $\gamma_1 = \mathcal{C}(O_1, r_1)$ and $\gamma_2 = \mathcal{C}(O_2, r_2)$ are tangent at P, then the centers O_1 and O_2 are distinct and the three points O_1, O_2, and P are collinear. Furthermore, the circles share a common tangent line at P.

T 8.2.2 (Circumscribed Circle): A triangle can be circumscribed if and only if the perpendicular bisectors of the sides of the triangle are concurrent. If a triangle can be circumscribed, then the circumcenter and the circumcircle are unique.

T 8.2.3: The Euclidean Parallel Postulate is equivalent to the assertion that every triangle can be cicumscribed.

T 8.2.4: If the Euclidean Parallel Postulate holds, then every triangle can be circumscribed.

C 8.2.5: In Euclidean geometry the three perpendicualar bisectors of the sides of any triangle are concurrent and meet at the circumcenter of the triangle.

T 8.2.8 (Inscribed Circle): Every triangle has a unique inscribed circle. The bisectors of interior angles in any triangle are concurrent and the point of concurrency is the incenter of the triangle.

T 8.2.12: Let γ be a cricle and P_1 a point on γ. For each $n \leq 3$ there is a regular polygon $P_1 P_2 \cdots P_n$ inscribed in γ.

T 8.3.1: Let $\triangle ABC$ be a triangle and let M be the midpoint of $\overline{AB}$. If $AM = MC$, then then $\angle ACB$ is a right angle.

C 8.3.2: If the vertices of triangle $\triangle ABC$ lie on a circle and $\overline{AB}$ is a diameter of that circle, then $\angle ACB$ is a right angle.

T 8.3.3: Let $\triangle ABC$ be a triangle and let M be the midpoint of $\overline{AB}$. If $\angle ACB$ is a right angle, then $AM = MC$.

C 8.3.4: If $\angle ACB$ is a right triangle, then $\overline{AB}$ is a diameter of the circle that circumscribes $\triangle ABC$.

T 8.3.5 (30-60-90): If the interior angles in triangle $\triangle ABC$ measure $30°$, $60°$, $90°$, then the length of the side opposite the $30°$ angle is one half the length of the hypotenuse.

T 8.3.6 (Converse 30-60-90): If $\triangle ABC$ is a right triangle such that the length of one leg is one-half the length of the hypotenuse, then the interior angles of the triangle measure $30°$, $60°$, and $90°$.

T 8.3.9 (Central Angle): The measure of an inscribed angle for a circle is one half the measure of the corresponding central angle.

C 8.3.10 (Inscribed Angle): If two inscribed angles intercept the same arc, then angles are congruent.

T 8.3.12: The power of a point is well defined; that is, the same value is obtained regardless of which line l is used in the definition as long as the line has at least one opint of intersection with the circle.

Transformations

T 10.1.6: The composition of two isometries is an isometry. The inverse of an isometry is an isometry.

T 10.1.7 (Properties of isometries): Let $T : \mathbb{P} \to \mathbb{P}$ be an isometry. Then T preserves;
1. collinearity; that is, if P, Q, and R are three collinear points, then $T(P)$, $T(Q)$, and $T(R)$ are colleinear,
2. betweenness of points; that is, if P, Q, and R are three points such that $P * Q * R$, then $T(P) * T(Q) * T(R)$,
3. segments; that is, if A and B are points and A' and B' are their images under T, then $T(\overline{AB}) = \overline{A'B'}$ and $\overline{A'B'} \cong \overline{AB}$,
4. lines; that is, if l is a line, then $T(l)$ is a line,
5. betweenness of rays; that is, if $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ are three rays such that $\overrightarrow{OQ}$ is between $\overrightarrow{OP}$ and $\overrightarrow{OR}$, then $\overrightarrow{O'Q'}$ is between $\overrightarrow{O'P'}$ and $\overrightarrow{O'R'}$,
6. angles; that is, if $\angle BAC$ is an angle, then $T(\angle BAC)$ is an angle and $T(\angle BAC) \cong \angle BAC$,
7. triangles; that is, if $\triangle BAC$ is a triangle, then $T(\triangle BAC)$ is a triangle and $T(\triangle BAC) \cong \triangle BAC$,
8. circles; that is, if γ is a circle with center O and radius r, then $T(\gamma)$ is a circle with center $T(O)$ and radius r,
9. areas; that is, if R is a polygonal region, then $T(R)$ is a polygonal region and $\alpha(T(R)) = \alpha(R)$.

T 10.1.8 (Existence Uniqueness Isometries): If $\triangle ABC$ and $\triangle DEF$ are two triangles with $\triangle ABC \cong \triangle DEF$, then there exists a unique isometry T such that $T(A) = D$, $T(B) = E$, and $T(C) = F$. Furthermore, T is the composition of either two or three reflections.

C 10.1.9: If f and g are two isometries and A, B, and C are three noncollinear points such that $f(A) = g(A)$, $f(B) = g(B)$, and $f(C) = g(C)$, then $f(P) = g(P)$ for every point in P.

L 10.1.10: An isometry that fixes three noncollinear points is the identity; that is, if A, B, and C are three noncollinear points and f is an isometry such that $f(A) = A$, $f(B) = B$, $f(C) = C$, then $f = \imath$.

C 10.1.11: Every isometry of the plane can be expressed as a composition of reflections. The number of reflections required is either two or three.

T 10.2.2 (Half-Turn): Let l and m be two lines that are perpendicular at O and let $h_O = \rho_m \circ \rho_l$ be the half-turn about O determined by these two lines;
1. if P is any point different from O, then O is the midpoint of the segment from P to $h_O(P)$,
2. if n and s are any two lines that are perpendicular at O, then $h_O = \rho_s \circ \rho_n = \rho_n \circ \rho_s$.

C 10.2.3: The point O is the only fixed point of h_O.

T 10.2.5 (Rotation): Let R_{AOB} be the rotation with center O and angle $\angle AOB$;
1. if P is any point different from O and $P'' = R_{AOB}(P)$, then $\mu(\angle POP'') = \mu(\angle AOB)$, **2.** if n is any line with $O \in n$, then there exist lines s and t such that $R_{AOB} = \rho_s \circ \rho_n = \rho_n \circ \rho_t$.

C 10.2.6: If $\mu(\angle AOB) \neq 0$, then O is the only fixed point of R_{AOB}.

T 10.2.8 (Translation): Let T_{AB} be a transition, where A and B are distinct points, and let $k = \overleftrightarrow{AB}$:
1. If P is a point on k, then $P' = T_{AB}(P)$ is the point on k such that $PP' = AB$ and $\overrightarrow{PP'}$ is equivalent to $\overrightarrow{AB}$. If P is a point not on k, then $P' = T_{AB}(P)$ is on the same side of k as P.
2. If n is any line that is perpendicular to k, then there exist lines s and t such that $R_{AOB} = \rho_s \circ \rho_n = \rho_n \circ \rho_t$.

C 10.2.9: If $A \neq B$, then T_{AB} has no fixed points.

T 10.3.2 (Glide Reflection): An isometry is a glide reflection if and only if it can be written as the composition of three reflections.

L 10.3.3: If l, m, and n are three lines that are concurrent at P, then there exists a line s such that P lies on s and $\rho_l \circ \rho_m = \rho_n \circ \rho_s$.

L 10.3.4: If l, m, and n are three lines that share a common perpendicular k, then there exists a line s such that $s \perp k$ and $\rho_l \circ \rho_m = \rho_n \circ \rho_s$.

L 10.3.5: If l, m, and n are three lines such that l and m intersect, then $\rho_l \circ \rho_m \circ \rho_n$ is a glide reflection.

L 10.3.6: If l, m, and n are three lines such that m and n intersect, then $\rho_l \circ \rho_m \circ \rho_n$ is a glide reflection.

T 10.3.7 (Classification Euclidean Motions): Every Euclidean motion is either the identity, a reflection, a half-turn, a rotation, a translation, or a glide reflection.

A 10.5.1 (Reflection Postulate): For every line l there exists a transformation $\rho_l : \mathbb{P} \to \mathbb{P}$, called the reflection in l, that satisfies the following conditions:
1. If P lies on l, then P is a fixed point for ρ_l.
2. If P lies in one of the half-planes determined by l, then $\rho_l(P)$ lies in the opposite half-plane.
3. ρ_l preserves collinearity, distance and angle measure.

T 10.5.5: The Reflection Postulate implies the Side-Angle-Side triangle congruence condition.

T 10.7.3 If $I_{O,r}$ is an inversion and P and Q are points that are not collinear with O, then $\triangle OPQ$ is similar to $\triangle OQ'P'$.

T 10.7.4 If $I_{O,r}$ is an iversion and l is a line that does not contain O, then $I_{O,r}(l \cup \{\infty\})$ is a circle that contains O.

C 10.7.5 If $I_{O,r}$ is an inversion and α is a circle such that $O \in$, then $I_{O,r}(\alpha - \{O\})$ is a line.

T 10.7.6 If l is a line and O lies on l, then $I_{O,r}(l \cup \{\infty\}) = l \cup \{\infty\}$.

T 10.7.7: If $I_{O,r}$ is an inversion and α is a circle that O does not lie on α, then $I_{O,r}(\alpha)$ is a circle.

T 10.7.9: If the circle β is orthogonal to $C(O,r)$, then $I_{O,r}(\beta) = \beta$.

T 10.7.10: Let $\mathcal{C} = C(O,r)$ and β be two circles. If there exists a point Q on β such that $Q' = I_{O,r}(Q)$ also lies on β and $Q' \neq Q$, then $\mathcal{C}$ is orthogonal to β.

C 10.7.11: Let $\mathcal{C} = C(O,r)$ and β be two circles. Then $\mathcal{C}$ is orthogonal to β if and only if $I_{O,r}(\beta) = \beta$.

C 10.7.12: If $\mathcal{C} = C(O,r)$ is a circle and R and S are two points inside $\mathcal{C}$ that do not lie on the same diameter of $\mathcal{C}$, then there exists a unique circle γ such that R and S both lie on γ and γ is orthogonal to $\mathcal{C}$.

T 10.7.13: Let $\mathcal{C} = C(O,r)$ be a circle and let P be a point inside $\mathcal{C}$ that is different from O. For every line t such that P lies on t but O does not lie on t, there exists a unique circle α such that t is tangent to α at P and α is orthogonal to $\mathcal{C}$.

T 10.7.15: If γ is a circle, P is a point on γ, O is a point not on γ, $I_{O,r}$ is an inversion, and C is a point on γ that is not antipodal to P, then the angle between $\gamma(P,C)$ and $\overrightarrow{PP'}$ is congruent to the angle between $\gamma'(P',C')$ and $\overrightarrow{P'P}$.

T 10.7.16: If l is a line that does not pass through O, $I_{O,r}$ is an inversion, γ' is the image of l under $I_{O,r}$, and P and C are two distinct points on l, then $\angle P'PC$ is congruent to the angle between $\gamma'(P',C')$ and $\overrightarrow{P'P}$.

T 10.7.17: If each of γ_1 and γ_2 is either a line or a circle, P is a point that lies on both γ_1 and γ_2, C_1 and C_2 are points on γ_1 and γ_2, and $I_{O,r}$ is an inversion, then the angle between $\gamma_1(P,C_1)$ and $\gamma_2(P,C_2)$ is congruent to the angle between $\gamma_1'(P',C_1')$ and $\gamma_2'(P',C_2')$.

T 10.7.19: If A, B, P, Q, and O are all distinct and A',B', P', and Q' are the images of A, B, P, Q under $I_{O,r}$, then $[AB,PQ] = [A'B',P'Q']$.

1. Kilder

[1] Gerard A. Venema, Foundaditons of Geometry Second Edition, MA2401 Geometry.