Summary - Geometri part 1

What is geometry?

- Geo = greek for earth
- metry = for the greek metrik for measurement
- Euclids elements

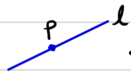
Why is Elements so important?

- First example of scientific work in an approximately modern sense
- Book is based on
  - A list of technical definitions
  - 5 postulates
  - 5 common notions
- Euclid deduced proofs of theorems and defined more terms
- Euclid recognized: it is not possible to define everything. One has to start somewhere

Axiomatic systems

- Undefined terms: technical terms that will be used for the subject
  - Examples: point, line...
- Defined terms/definitions: using the undefined (and previously defined) terms to define new terms
- Axioms/postulates: statements that are accepted without a proof. Everything else in the system should be logically deduced from them
- Theorems and proofs: logical consequences of the axioms
- Interpretations and models: An interpretation of an axiomatic system: a particular way to give meaning to the undefined terms in that system. An interpretation is called a model for the axiomatic system if the axioms are correct/true statements in that interpretation.
- Consistence: Axioms are said to be consistent if no logical contradiction can be derived from them

Example: Incidence Geometry

- Undefined terms:
  - point  ← P lies on l
  - line
  - lie on ("point P lies on line l) or incident (point P is incident with line l)

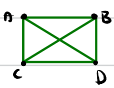
  (Definitions: collinear, noncollinear)

- Incidence axioms:

1. For every pair of distinct points P and Q, there exists exactly one line l such that both P and Q lie on l.
2. For every line l there exists at least two distinct points P and Q such that both P and Q lie on l.
3. There exist three noncollinear points

One example: Four point geometry

- Points: Symbols A, B, C, D
- Lines: sets of two points {A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}
- Lie on: "is an element of"



F.ex: A & D are both elements of the set {A,D}, because they both lie on the line $\overline{AD}$

Axiom 1: Always one line between two points

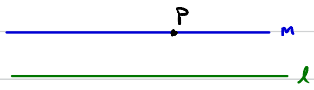
Axiom 2: Always two points on a line

Axiom 3: Only two points per line, therefore three noncollinear points (A is not on $\overline{CD}$)

No sets contain three points

Parallel lines

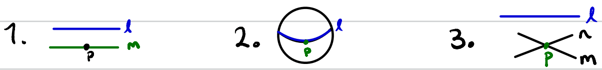
- **Parallel**: two lines ℓ and m are parallel if there is no point P such that P lies both on ℓ and m (Notation: $\ell \parallel m$)



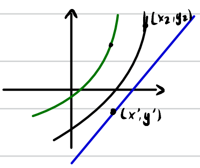
- **The parallel postulates**

For every line ℓ and for every point P that does not lie on ℓ ...

1. ... there is exactly one line m such that $P \in m$ and $\ell \parallel m$ (Euclidian parallel postulate)
2. ... there is no line m such that $P \in m$ and $\ell \parallel m$ (Elliptic Parallel postulate)
3. ... there are at least two lines m and n such that $P \in m$, $P \in n$ and $\ell \parallel m$, $\ell \parallel n$ (Hyperbolic parallel postulate)

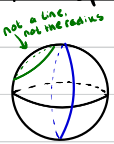


The cartesian plane



$ax + by + c = 0$ All axioms are fulfilled

The sphere



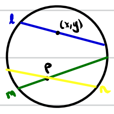
All points are on the surface of the sphere, and all lines are circles on the sphere

All lines have the radius of the sphere

The lines always intersect = no parallel lines

Not following the axioms of incidence geometry, infinetly circles through two points

The klein disk



(x,y), $x^2+y^2<1$ (only lines inside the disk)

$\ell \parallel m$, does not intersect inside the klein disk.

Hyperbolic parallel postulate

All axioms fulfilled

Axiomatic systems ⟷ Parallel postulates

We have now seen some examples of models of axiomatic systems and connected them to different parallel postulates:

- Cartesian plane: Euclidian parallel postulate
- The sphere: Elliptic parallel postulate
- Klein disk: Hyperbolic parallel postulate

Schematic overview

Axioms 1-6 + Axiom 7

Axioms NG1-NG6 (Neutral Geometry)

+ Euclidian Parallel Postulate	+ Elliptic Parallel Postulate	+ Hyperbolic Parallel Postulate
↓	↓	↓
Euclidian Geometry	No consistent model	Hyperbolic Geometry

NOYTRAL GEOMETRI

Plane geometry / Neutral Geometry

- Undefined terms in Plane Geometry:
  - Point
  - line
  - distance
  - half-plane
  - angle measure

> We will state 6 axioms for the neutral geometry.
>
> Axioms 1-5 corresponds to the 5 undefined terms and "describe" their properties

Axioms for neutral geometry - NG1

The existence postulate (3.1.1)

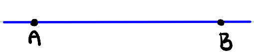
- The collection of all points forms a nonempty set. There is more than one point in that set.

(There are more than one point)

Axioms for neutral geometry - NG2

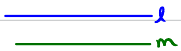
The Incidence postulate (3.1.3)

- Every line is a set of points. For each pair of distinct points A and B there is exactly one line ℓ such that $A \in \ell$ and $B \in \ell$

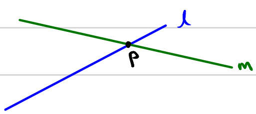


(For every pair of distinct points, there is exactly one line through these points)

- Definition : parallel lines
  - Two lines ℓ, m parallel $(\ell \parallel m) \Leftrightarrow$ there is no point in common



- Theorem (2.6.2, 3.1.7)
  - If ℓ and m are two distinct, nonparallel lines, then there exists exactly one point P such that P lies on both ℓ and m



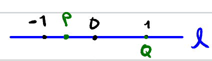
Axioms for neutral geometry - NG3

The ruler postulate (3.2.1)

- For every pair of points P and Q there exists a real number PQ, called the distance from P to Q. For each line ℓ there is a one-to-one correspondence from ℓ to $\mathbb{R}$ such that if P and Q are points on the line that correspond to the real numbers x and y, respectively, then $PQ = |x-y|$.

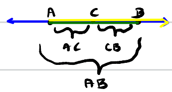
(One can measure the distance of each pair of points on a line using real numbers. There is a one-to-one correspondence between the line and $\mathbb{R}$.

- Specifies basic properties of distance measurements
- Implies that it is possible to introduce (real number) coordinates on a line
- Implies that lines are continous (with no gaps))



$P = -0.5$
$Q = 1$
$\Big\} PQ = |x-y| = |-0.5-1| = |-1.5| = \underline{1.5}$

- **Definition**: between (3.2.2), segment, ray (3.2.4)
  - C is between A and B



$A * C * B$

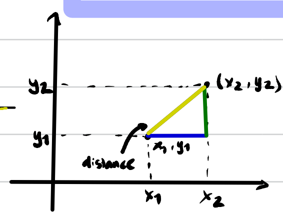
— $\overline{AB} = \{A, B\} \cup \{P \mid A * P * B\}$ (segment)

— $\overrightarrow{AB} = \overline{AB} \cup \{P \mid A * B * P\}$ (ray)

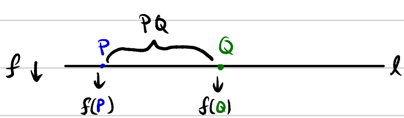
Note: $\overline{AB} \cong \overline{CD}$, then $AB = CD$ (distance)

- **Definition**: metric (3.2.9)
  - A metric is a function $D: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ such that
    1) $D(P,Q) = D(Q,P)$ for every two points P and Q (metric is symmetric)
    2) $D(P,Q) \geq 0$ for every two points P and Q (metric is positive)
    3) $D(P,Q) = 0$ if and only if $P = Q$
  - Examples: Euclidian metric, taxicab metric
  - Euclidian metric: $d((x_1, y_1), (x_2, y_2)) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$
  - Taxicab metric: $\rho((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$

> Metrics are used to describe what distance means in a geometry

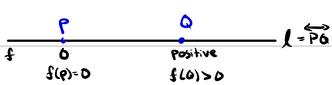


- Coordinate function 3.2.13:
  - Let ℓ be a line. A one-to-one correspondence $f: \ell \to \mathbb{R}$ such that $PQ = |f(P) - f(Q)|$ for every P and Q on ℓ is called a coordinate function for the line ℓ, and the number $f(P)$ is called the coordinate of the point P.

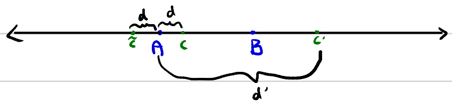


$|f(P) - f(Q)| = PQ$

- **The ruler placement "postulate"** (3.2.16)
  - For every pair of distinct points P and Q, there is a coordinate function $f: \overleftrightarrow{PQ} \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$

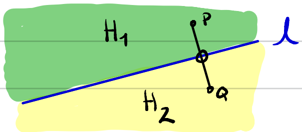


- The point construction "postulate" (3.2.23)
  - If A and B are distinct points and d is any nonnegative number, then there exists a unique point C such that C lies on $\overrightarrow{AB}$ and $AC = d$

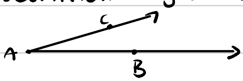


Axioms for Neutral Geometry - N64

- The Plane Separation Postulate (3.3.2)
  - For every line ℓ, the points that do not lie on ℓ form two disjoint, nonempty sets H_1 and H_2, called half-planes bounded by ℓ, such that the following conditions are satisfied:
    - Each of H_1 and H_2 is convex
    - If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ}$ intersects ℓ.
  - (Every line divides the plane into two convex sets, the halfplanes. A connection between two points, one in each halfplane, intersects the line).



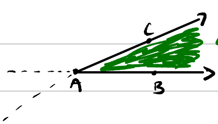
- Definition: angle (3.3.6)



$\overrightarrow{AB} \cup \overrightarrow{AC} = \measuredangle BAC$ or $\measuredangle CAB$

sides of the angle $\measuredangle BAC$

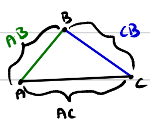
- Definition: interior of an angle (3.3.7)



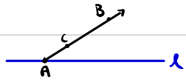
interior of the $\measuredangle CAB$

- Definition: Triangle (3.3.11)

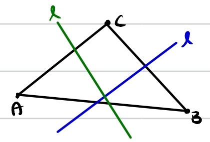
$\triangle ABC = \overline{AB} \cup \overline{AC} \cup \overline{BC}$, A, B, C noncollinear



- The ray theorem (3.3.9)
  - Let ℓ be a line, A a point on ℓ, and B an external point for ℓ. If C is on $\overrightarrow{AB}$ and $C \neq A$, then B and C are on the same side of ℓ

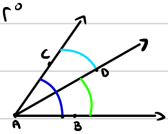


- Pasch's axiom (3.3.12)
  - Let $\triangle ABC$ be a triangle and let ℓ be a line such that none of A, B and C lies on ℓ. If ℓ intersects $\overline{AB}$, then ℓ also intersects either $\overline{AC}$ or $\overline{BC}$



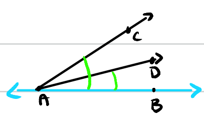
Axioms for Neutral Geometry - NG5

- The Protractor Postulate (3.4.1)
  - For every angle $\angle BAC$ there is a real number $\mu(\angle BAC)$, called the measure of $\angle BAC$, such that the following conditions are satisfied:
    1) $0^\circ \leq \mu(\angle BAC) < 180^\circ$ for every angle $\angle BAC$
    2) $\mu(\angle BAC) = 0^\circ$ if and only if $\overrightarrow{AB} = \overrightarrow{AC}$
    3) Angle Construction Postulate: for each real number r, $0 < r < 180$, and for each half-plane H bounded by $\overleftrightarrow{AB}$ there exists a unique ray $\overrightarrow{AE}$ such that E is in H and $\mu(\angle BAE) = r^\circ$
    4) Angle Addition Postulate: If the ray $\overrightarrow{AD}$ is between the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ then $\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)$

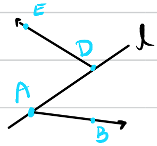


(we can measure and construct angles ($\leq 0^\circ$ and $< 180^\circ$) and add angles if they share a ray)

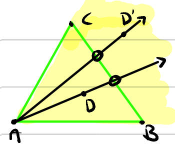
- Betweenness Theorem for Rays (3.4.5)
  - Let A, B, C and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $\mu(\angle BAD) < \mu(\angle BAC)$ if and only if $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$



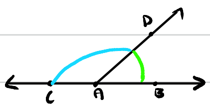
- The Z-theorem (3.5.1)
  - Let ℓ be a line and let A and D be distinct points on ℓ. If B and E are points on opposite sides of ℓ, then $\overrightarrow{AB} \cap \overrightarrow{DE} = \emptyset$



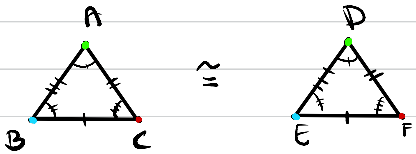
- The crossbar theorem
  - If $\triangle ABC$ is a triangle and D is a point in the interior of $\angle BAC$, then there is a point G such that G lies both on $\overrightarrow{AD}$ and $\overline{BC}$.



- The linear pair theorem
  - If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180°$



- Definition: Congruence of triangles (3.6.1)



The triangles must be exactly the same.

Sides & angles are the same size

$\triangle ABC \cong \triangle DEF$

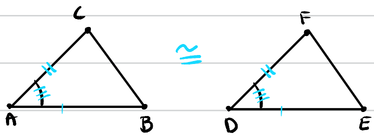
congruence

Axioms for Neutral Geometry - NG 6

The Side-Angle-Side postulate (3.6.3)

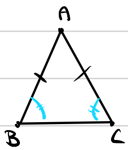
- If ΔABC and ΔDEF are two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\Delta ABC \cong \Delta DEF$.

(Triangles are congruent if two sides and the angle between them are congruent)



- Isosceles Triangle Theorem (3.6.5)
  - If ΔABC is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.

  Actually (which is proven in Chapter 4.2), the converse also holds:
  - If ΔABC is a triangle and $\angle ABC \cong \angle ACB$, then $\overline{AB} \cong \overline{AC}$ (4.2.2)



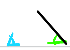
if two sides are congruent, so will the angles be (and opposite)

More about triangles

- Congruences

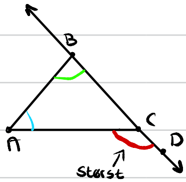
  Based on the side-angle-side postulate (SAS), we can prove more congruences about triangles:

  Two triangles are congruent if they have congruences for one of the following combinations

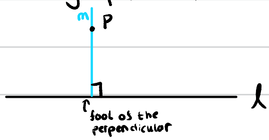
  - angle-side-angle (ASA), 4.2.1 
  - angle-angle-side (AAS), 4.2.3 
  - Side-Side-Side (SSS), 4.2.7 

- Note: Side-Side-angle is in general not sufficient for congruence

- Exterior angle theorem (4.1.2)
  - If ΔABC is a triangle and D is a point such that $\overrightarrow{CD}$ is opposite to $\overrightarrow{CB}$, then $m(\angle DCA) > m(\angle BAC)$ and $m(\angle DCA) > m(\angle ABC)$
  - Consequence: triangles cannot have two right angles

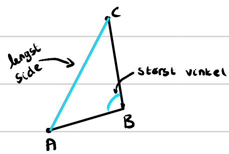


- Theorem on existence and uniqueness of perpendiculars
  - For every line ℓ and for every point P, there exists a unique line m such that P lies on m and $m \perp \ell$

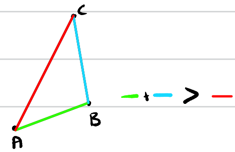


$\perp$ = perpendicular

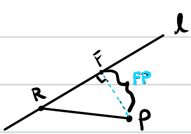
- Theorem: Scalene Inequality (4.3.1)
  - In any triangle, the greater side lies opposite the greater angle and the greater angle lies opposite the greater side



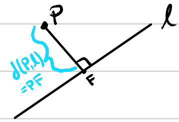
- Theorem: Triangle Inequality (4.3.2)
  - If A, B and C are three noncollinear points, then $AC < AB + BC$



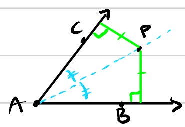
- Theorem: Distance of point and line (4.3.4)
  - Let ℓ be a line, let P be an external point, and let F be the foot of the perpendicular from P to ℓ. If R is any point on ℓ that is different from F, then $PR > PF$



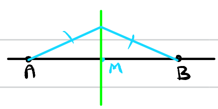
- **Definition: distance from point to line (4.3.5)**
  - If ℓ is a line and P is a point, then the distance from P to ℓ, denoted $d(P,\ell)$, is defined to be the distance from P to the foot point of the perpendicular from P to ℓ.



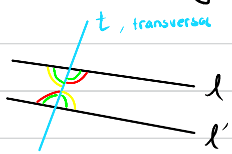
- **Theorem: Pointwise Characterization of Angle Bisector (4.3.6)**
  - Let A, B and C be three noncollinear points and let P be a point in the interior of $\angle BAC$ if and only if $d(P,\overleftrightarrow{AB}) = d(P,\overleftrightarrow{AC})$



- **Theorem: Pointwise Characterization of Perpendicular Bisector**
  - Let A and B be distinct points. A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$



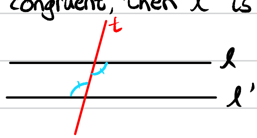
- **Definition: transversal, interior angle, alternate interior angle (4.4.1)**



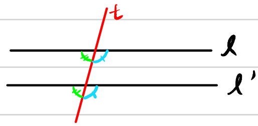
$\measuredangle$ = interior angle

$\measuredangle$ $\measuredangle$ = alternate interior angle (two pairs)

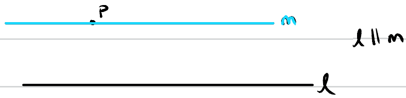
- **Alternate Interior Angle Theorem**
  - If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'



- Corresponding angles theorem (4.4.4)
  - If ℓ and ℓ' are two lines cut by a transversal t in such a way that two corresponding angles are congruent, then ℓ is parallel to ℓ'.



- Corollary: Existence of Parallels (4.4.6)
  - If ℓ is a line and P is an external point, then there is a line m such that P lies on m and m is parallel to ℓ.



=> Elliptic parallel postulate ↯

Continuing Neutral geometry, but not the axiom N6...

- Saccheri-Legendre theorem (4.5.2)

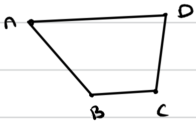
If $\triangle ABC$ is any triangle, then $\sigma(\triangle ABC) \leq 180°$

In other words → Not always exactly 180° (can be smaller)

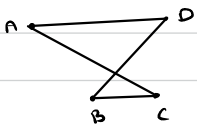
- Theorem 4.6.4

If $\square ABCD$ is a convex quadrilateral, then $\sigma(\square ABCD) \leq 360°$

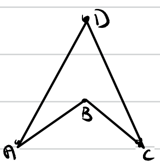
Same as the theorem concerning triangles



quadrilateral

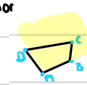


not quadrilateral



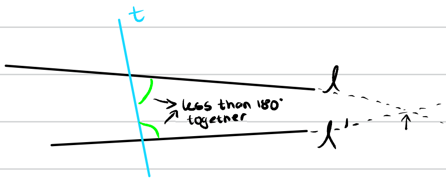
quadrilateral
But not convex

Convex = the last point is included in the interior of the other points' angle. F.eks:

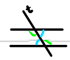


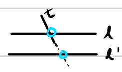
C is in the interior of $\angle DAB$

- Euclid's postulate V
  - If ℓ and ℓ' are two lines cut by a transversal t in such a way that the sum of the measures of the two interior angles on one side of t is less than 180°, then ℓ and ℓ' intersect on that side of t.



The following are equivalent:

- The Euclidian Parallel Postulate
- Euclid's postulate V (proof of equivalence: 4.7.2)
- The converse of the alternate interior angle theorem (4.7.2) 

  - If two parallel lines are cut by a transversal, both pairs of alternate interior angles are congruent
- Angle sum postulates
  - $\sigma(\triangle ABC) = 180°$ for all triangles $\triangle ABC$
- 4.8.4
  - Part 1: $\sigma(\triangle ABC) = 180°$ for one triangles $\triangle ABC$
  - Part 3: There exists a rectangle $\square ABCD$
- Proclus' Axiom (4.7.3, part 1) 

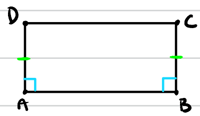
  - If ℓ and ℓ' are parallel lines and $t \neq \ell$ is a line such that $t \cap \ell \neq \emptyset$, then $t \cap \ell' \neq \emptyset$
- Transitivity of parallelism (4.7.3, part 4) 

$\ell \parallel n$
  - If ℓ is parallel to m and m is parallel to n, then either $\ell = n$ or $\ell \parallel n$
- Every triangle can be circumscribed (8.2.3) 

- Defect of triangles and quadrilateral (4.8.1)
  - Defect of $\triangle ABC$: $\delta(\triangle ABC) = 180° - \sigma(\triangle ABC)$

    By the Saccheri-Legendre Theorem, the defect of every triangle is nonnegative.
  - Defect of a convex quadrilateral $\square ABCD$: $\delta(\square ABCD) = 360° - \sigma(\square ABCD)$

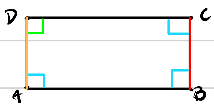
- Saccheri quadrilateral (4.8.8)
  - A quadrilateral $\square ABCD$ such that $\angle ABC$ and $\angle DAB$ are right angles and $\overline{AD} \cong \overline{BC}$



Properties

1. $\overline{AC} \cong \overline{BD}$
2. $\angle BCD \cong \angle ADC$
3. If $AE = EB$ and $DF = FC$ then $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{EF} \perp \overleftrightarrow{DC}$
4. $\square ABCD$ is a parallelogram
5. $\square ABCD$ is convex
6. $\angle BCD$ and $\angle ADC$ are right or acute angles

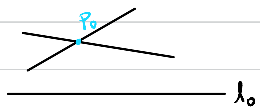
- Lambert quadrilateral
  - A quadrilateral in which three of the angles are right angles



Properties

1. $\square ABCD$ is a parallelogram
2. $\square ABCD$ is convex
3. $\angle ADC$ is either right or acute angles
4. $BC \leq AD$.

- The universal Hyperbolic Theorem
  - If there exists one line l_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to l_0, then for every line l and for every external point P there exist at least two lines that pass through P and are parallel to l.

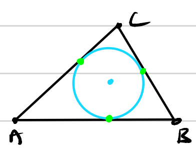


  - Corollary (4.9.2): The hyperbolic parallel postulate is equivalent to the negation of the Euclidian Parallel Postulate

  - Corollary (4.9.3): In any model of neutral geometry, either the Euclidian parallel postulate or the hyperbolic parallel postulate will hold

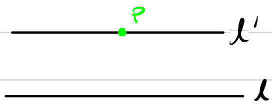
Circles in Neutral Geometry

- Inscribed Circle Theorem (8.2.8)
  - Every triangle has a unique inscribed circle

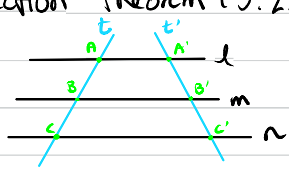


Euclidian Geometry

- Euclidian Parallel Postulate is assumed as the 7th Axiom (together with the 6 axioms from neutral geometry)



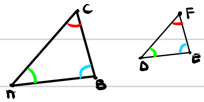
- Parallel projection Theorem (5.2.1)



$A*B*C$

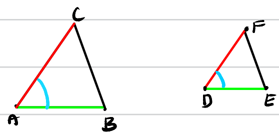
$$\frac{AB}{AC} = \frac{A'B'}{A'C'}$$

- Fundamental Theorem on Similar Triangles (5.3.1)
  - If ΔABC and ΔDEF are two triangles such that $\Delta ABC \sim \Delta DEF$, then $\frac{AB}{AC} = \frac{DE}{DF}$.



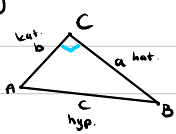
  - Corollary (5.3.2): If ΔABC and ΔDEF are two triangles such that $\Delta ABC \sim \Delta DEF$, then there is a positive number r such that $DE = r \cdot AB$, $DF = r \cdot AC$, and $EF = r \cdot BC$.

- SAS Similarity Criterion (5.3.3)
  - If ΔABC and ΔDEF are two triangles such that $\angle CAB \cong \angle FDE$ and $\frac{AB}{AC} = \frac{DE}{DF}$, then $\Delta ABC \sim \Delta DEF$



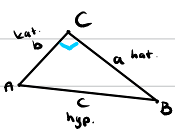
- Converse to Similar Triangles Theorem (5.3.4)
  - ΔABC and ΔDEF with $\underbrace{\frac{AB}{DE}}_{r} = \underbrace{\frac{AC}{DF}}_{r} = \underbrace{\frac{BC}{EF}}_{r} \Rightarrow \Delta ABC \sim \Delta DEF$

Pythagorean Theorem (5.4.1)



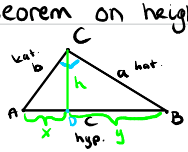
$\Rightarrow a^2 + b^2 = c^2$

- Converse to the pythagorean theorem (5.4.5)



$\Leftarrow a^2 + b^2 = c^2$

- Theorem on height of triangle (5.4.3)

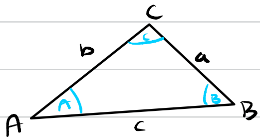


$h = \sqrt{x \cdot y}$

Law of Sines and Cosines (5.5.3, 5.5.4)

- Pythagorean Identity : θ any angle
  - $\sin^2\theta + \cos^2\theta = 1$

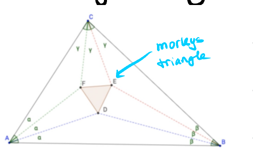
- Law of sines :
  - $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

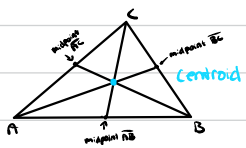


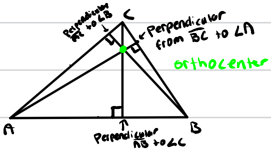
- Law of cosines:
  - $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos C$

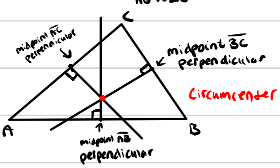
Morleys Theorem (Excursus)

- The point of intersection of the adjacent angle trisectors of the angles of any triangle ΔABC are the polygon vertices of an equilateral triangle ΔDEF known as the Morley triangle









Definition of points:

- Centroid : point where the three medians of the triangle meet (A median of a triangle is a line segment from one vertex to the midpoint on the opposite side of the triangle)
- Orthocenter: The intersection of the three altitudes of a triangle
- Circumcenter: the point where all the perpendicular bisectors of the triangle's sides meet

Euler line and Nine-point circle

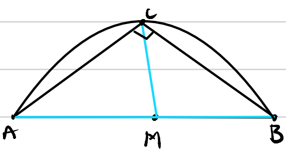
- Euler line: the line on which the centroid, the orthocenter and the circumcenter of a triangle lie

- Nine-point circle: Circle that can be constructed for any given triangle. The following nine points all lie on the circle:
  - The midpoint of each side of the triangle (3 punkt)
  - The foot of each altitude (3 punkt)
  - The midpoint of the line segment from each vertex of the triangle to the orthocenter (3 punkt)

Theorems on circles in Euclidian Geometry

- Theorem (8.3.1 and 8.3.3 together)

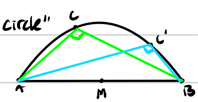
  Let ΔABC be a triangle and let M be the midpoint of $\overline{AB}$. If and only if $AM = MC$, then $\angle ACB$ is a right angle



- Corollary (8.3.2)

  If the vertices of a triangle ΔABC lie on a circle and $\overline{AB}$ is a diameter of that circle, then $\angle ACB$ is a right angle

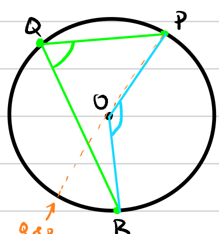
"Thales circle"



Definitions (8.3.7 & 8.3.8)

- Inscribed angle $\angle PQR$
- central angle $\angle POR$
- corresponding angle Central angle = $\frac{\text{inscribed angle}}{2}$

central angle theorem



$y = C(0,r)$

Q & R on opposite sides

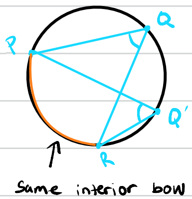
$\angle POR$: corresponding central angle for $\angle PQR$

Central angle theorem (8.3.9)

- The measure of an inscribed angle for a circle is one half the measure of the corresponding central angle

Inscribed angle theorem (8.3.10)

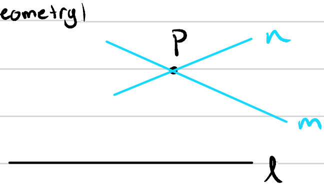
- If two inscribed angles intercept the same arc, then the angles are congruent



$\angle PQR \cong \angle PQ'R$ if they share the same interior bow (marked in orange)

HYPERBOLIC GEOMETRY (H.G.)

- Hyperbolic Parallel Postulate is assumed as the 7th Axiom (together with the 6 axioms from Neutral Geometry)



$m \parallel l$, $n \parallel l$

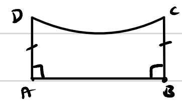
- Angle sum and defect in H.G.
  - Theorem (6.1.1) : For every triangle ΔABC, $\sigma(\Delta ABC) < 180^\circ$

- Corollary (6.1.2) : For every triangle ΔABC, $0^\circ < \delta(\Delta ABC) < 180^\circ$

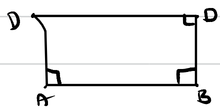
- Theorem (6.1.3) : For every convex quadrilateral $\square ABCD$, $\sigma(\square ABCD) < 360^\circ$

Properties of quadrilaterals in H. G.

- Corollary (6.1.4): The summit angles in Saccheri quadrilaterals are acute



- Corollary (6.1.5): The fourth angle in a Lambert quadrilateral is acute

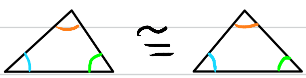


- Theorem (6.1.6): There does not exist a rectangle

- Theorem (6.1.7): In a Lambert quadrilateral, the length of a side between two right angles is strictly less than the length of the opposite side.

AAA congruence criterium (6.1.11)

- If ΔABC is similar to ΔDEF, then ΔABC is congruent to ΔDEF

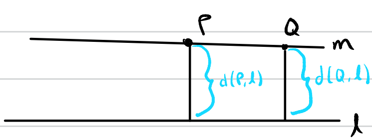


(congruent in H.G., not in Euclidian)

Parallel lines in H. G

- Theorem (6.2.1)

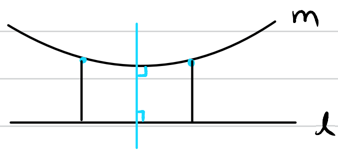
- If ℓ is a line, P is an external, and m is a line such that P lies on m, then there exists at most one point Q such that $P \neq Q$, Q lies on m, and $d(Q,\ell) = d(P,\ell)$



$d(P,\ell) = d(Q,\ell)$

Theorem (6.2.3):

- If l and m are parallel lines and there exist two points on m that are equidistant from l, then l and m admit a common perpendicular



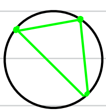
Theorem (6.2.4):

- If lines l and m admit a common perpendicular, then that common perpendicular is unique.

Circumscribed triangle

- Theorem (8.2.6):

  If the Euclidian Parallel Postulates fails, then there exists a triangle that cannot be circumscribed



← this can be allowed in H.G as well

= That means not all triangles cannot be circumscribed, it just means at <u>least one</u> cannot.

Also repeat for the exam:

- The cartesian model for Euclidian Geometry (lecture 26, chapter 11.1)
- The Poincaré disk model (lecture 27, chapter 11.2)