Solutions to Selected Exercises

in

Complex Analysis with Applications

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Solutions to Exercises 1.1

1. We have

$$\frac{1-i}{2} = \frac{1}{2} + (-\frac{1}{2})i.$$

So $a = \dfrac{1}{2}$ and $b = -\dfrac{1}{2}$.

5. We have

$$\begin{aligned}(\overline{2-i})^2 &= (2+i)^2 \quad (\text{because } \overline{2} - \overline{i} = 2-(-i) = 2+i)\\ &= 4 + 4i + \overbrace{(i)^2}^{=-1} = 3+4i.\end{aligned}$$

So $a = 3$ and $b = 4$.

9. We have

$$\left(\frac{1}{2}+\frac{i}{7}\right)\left(\frac{3}{2}-i\right) = \frac{1}{2}\frac{3}{2} + i\left(\frac{3}{14}-\frac{1}{2}\right)\overbrace{-i\,\frac{i}{7}}^{=\frac{1}{7}} = \frac{25}{28} - i\frac{2}{7}$$

So $a = \frac{25}{28}$ and $b = -\frac{2}{7}$.

13. Multiplying and dividing by the conjugate of the denominator, i.e. by $\overline{2-i} = 2+i$ we get

$$\begin{aligned}\frac{14+13i}{2-i} &= \frac{(14+13i)(2+i)}{(2-i)(2+i)} = \frac{14\cdot 2 + 14\cdot i + 13i\cdot 2 + 13i^2}{4+1}\\ &= \frac{28+14i+26i-13}{5} = \frac{15}{5} + \frac{40}{5}i = 3+8i\end{aligned}$$

So $a = 3$ and $b = 8$.

17. Multiplying and dividing by the conjugate of the denominator, i.e. by $\overline{x-iy} = x+iy$ we get

$$\begin{aligned}\frac{x+iy}{x-iy} &= \frac{x+iy}{x-iy}\cdot\frac{(x+iy)}{(x+iy)}\\ &= \frac{(x+iy)^2}{x^2+y^2}\\ &= \frac{x^2+2xyi+y^2i^2}{x^2+y^2}\\ &= \frac{x^2-y^2+2xyi}{x^2+y^2}\\ &= \frac{x^2-y^2}{x^2+y^2} + \frac{2xy}{x^2+y^2}i.\end{aligned}$$

So $a = \dfrac{x^2 - y^2}{x^2 + y^2}$ and $b = \dfrac{2xy}{x^2 + y^2}$.

21. Let $z_j = x_j + i\, y_j$, where x_j, y_j are real numbers and $j = 1, 2, 3$.
(a) We have

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2) = (x_2 + x_1) + i\,(y_2 + y_1) = z_2 + z_1.$$

(b) We have

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1) = (x_2 x_1 - y_2 y_1) + i\,(y_2 x_1 + y_1 x_2) = z_2 z_1.$$

(c) The associative property

$$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$$

is valid for real numbers and so it holds for the real and imaginary parts of $z_,1 z_2, z_3$. Consequently we have

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(d) Note that

$$\begin{aligned}(z_1 z_2) z_3 &= \big((x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)\big)(x_3 + i\, y_3)\\ &= (x_1 x_2 - y_1 y_2) x_3 - (x_1 y_2 + x_2 y_1) y_3 + i\,\big((x_1 x_2 - y_1 y_2) y_3 + (x_1 y_2 + x_2 y_1) x_3\big)\end{aligned}$$

and also that

$$\begin{aligned}z_1(z_2 z_3) &= (x_1 + i\, y_1)\big((x_2 x_3 - y_2 y_3) + i\,(x_2 y_3 + x_3 y_2)\big)\\ &= x_1(x_2 x_3 - y_2 y_3) - y_1(x_2 y_3 + x_3 y_2) + i\,\big(x_1(x_2 y_3 + x_3 y_2) + y_1(x_2 x_3 - y_2 y_3)\big).\end{aligned}$$

We simply check now that the real and imaginary parts of the complex numbers $(z_1 z_2) z_3$ and $z_1(z_2 z_3)$ coincide. Thus $(z_1 z_2) z_3 = z_1(z_2 z_3)$.
(e) Notice that

$$\begin{aligned}z_1(z_2 + z_3) &= (x_1 + i\, y_1)(x_2 + y_2 + i\,(x_3 + y_3))\\ &= x_1(x_2 + y_2) - y_1(x_3 + y_3) + i\,\big(x_1(x_3 + y_3) + y_1(x_2 + y_2)\big)\end{aligned}$$

while

$$\begin{aligned}z_1 z_2 + z_1 z_3 &= (x_1 + i\, y_1)(x_2 + i\, y_2) + (x_1 + i\, y_1)(x_3 + i\, y_3))\\ &= x_1 x_2 - y_1 y_2 + i\,(x_1 y_2 + x_2 y_1) + x_1 x_3 - y_1 y_3 + i\,(x_1 y_3 + x_3 y_1).\end{aligned}$$

The real and imaginary parts of these numbers are equal so the distributive property

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

holds.

25.

$$\begin{aligned}
(2+3i)z &= (2-i)z - i\\
\{(2+3i)-(2-i)\}z &= -i\\
(2+3i-2+i)z &= -i\\
4iz &= -i\\
z &= -\frac{i}{4i}\\
z &= -\frac{1}{4}
\end{aligned}$$

So $z = -\dfrac{1}{4}$.

29.

$$\begin{aligned}
\overline{iz+2i} &= 4\\
\overline{\overline{iz+2i}} &= \overline{4} && \text{(Conjugating both sides)}\\
iz+2i &= 4 && \text{(Using problem 34)}\\
iz &= 4-2i\\
z &= \frac{4-2i}{i}\\
z &= \frac{4-2i}{i}\cdot\left(\frac{-i}{-i}\right)\\
z &= \frac{-4i+2i^2}{1}\\
z &= -2-4i
\end{aligned}$$

So $z = -2-4i$.

33. We are given

$$\begin{aligned}
(1-i)z_1 + z_2 &= 3+2i\\
z_1 + (2-i)z_2 &= 2+i
\end{aligned}$$

From the first equation we obtain

$$\begin{aligned}
z_2 &= 3+2i-(1-i)z_1\\
&= 3+2i-z_1+iz_1
\end{aligned}$$

We then substitute the expression for z_2 into the second equation to get

$$\begin{aligned}
z_1 + (2-i)(3+2i-z_1+iz_1) &= 2+i \\
z_1 + 6 + 4i - 2z_1 + 2iz_1 - 3i - 2i^2 + iz_1 - i^2 z_1 &= 2+i \\
z_1 + 6 + 4i - 2z_1 + 2iz_1 - 3i + 2 + iz_1 + z_1 &= 2+i \\
3iz_1 + 8 + i &= 2+i \\
3iz_1 &= 2+i-8-i \\
3iz_1 &= -6 \\
z_1 &= \frac{-6}{3i} \\
z_1 &= -2i
\end{aligned}$$

Thus

$$\begin{aligned}
z_2 &= 3 + 2i - 2i + 2i^2 \\
&= 3 + 2i - 2i - 2 \\
&= 1
\end{aligned}$$

So $z_1 = 2i$ and $z_2 = 1$.

37. We have

$$\begin{aligned}
x^2 + 4x + 5 &= 0 \\
x &= \frac{-4 \pm \sqrt{4^2 - 4\times 5}}{2} \qquad \text{(By the quadratic formula)} \\
&= \frac{-4 \pm \sqrt{-4}}{2} \\
&= \frac{-4 \pm 2i}{2} \\
&= -2 \pm i
\end{aligned}$$

41. We have

$$\begin{aligned}
\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} &= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} \\
&= \overline{a_n} \cdot \overline{z^n} + \overline{a_{n-1}} \cdot \overline{z^{n-1}} + \cdots + \overline{a_1} \cdot \overline{z} + \overline{a_0} \\
&\quad \text{[Since } a_0, a_1, \cdots, a_{n-1}, a_n \text{ are all real]} \\
&= a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \cdots + a_1 \overline{z} + a_0 \\
&\quad \text{[by using the property that } \overline{z^n} = (\overline{z})^n\text{]} \\
&= a_n (\overline{z})^n + a_{n-1} (\overline{z})^{n-1} + \cdots + a_1 (\overline{z}) + a_0.
\end{aligned}$$

45. We already know that $z = 1 + i$ is a root of $p(z) = z^4 + 4$, then then by problem 48, $\overline{z} = 1 - i$ is also a root of $p(z)$. Thus $h(z) = (z - 1 - i)(z - 1 + i) = z^2 - 2z + 2$ divides $p(z)$

and we get

$$\begin{aligned} p(z) &= z^4 + 4 \\ &= (z^2 - 2z + 2)(z^2 + 2z + 2) \\ &= (z - 1 - i)(z - 1 + i)(z + 1 - i)(z + 1 + i) \end{aligned}$$

Hence, $p(z) = (z-1-i)(z-1+i)(z+1-i)(z+1+i)$ and its roots are $1+i, 1-i, -1+i, -1-i$.

49. Let $z = x + iy$ such that $z^2 = -3 + 4i$. We have

$$\begin{aligned} (x + iy)^2 &= -3 + 4i \\ (x^2 - y^2) + 2xyi &= -3 + 4i \end{aligned}$$

By comparing the real and imaginary parts we get the following equations

$$\begin{aligned} x^2 - y^2 &= -3 \\ 2xy &= 4 \end{aligned}$$

From the first equation we have

$$y^2 = x^2 + 3$$

We now consider the second equation and substitute the expression for y^2

$$\begin{aligned} 2xy &= 4 \\ xy &= 2 \\ x^2y^2 &= 4 && \text{(Squaring both sides)} \\ x^2(x^2 + 3) &= 4 && \text{(Substituting the expression for } y^2\text{)} \\ x^4 + 3x^2 &= 4 \\ x^4 + 3x^2 - 4 &= 0 \\ u^2 + 3u - 4 &= 0 && \text{(Put } u = x^2\text{)} \\ (u + 4)(u - 1) &= 0 \\ u &= 1 && \text{(Negative root is discarded since } u = x^2 \text{ is non-negative)} \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

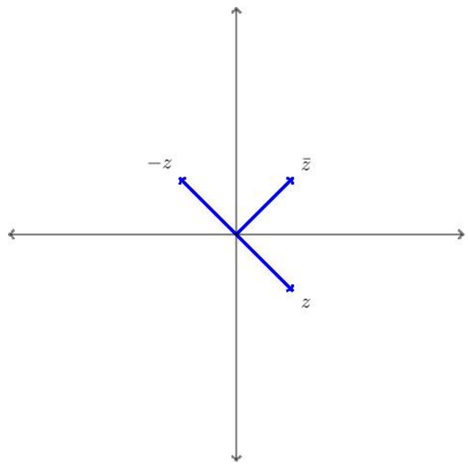
Now from the relation $xy = 2$, we compute the value of y

$$\begin{aligned} y &= \frac{2}{x} \\ &= \pm 2 \end{aligned}$$

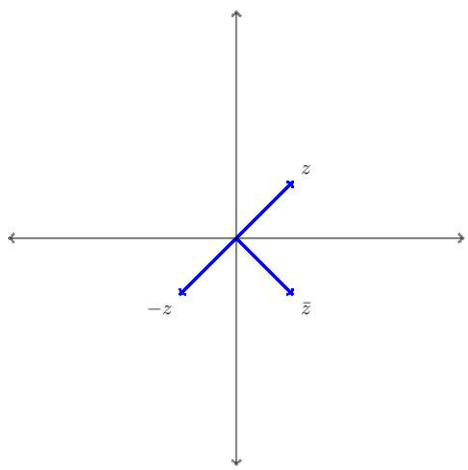
We thus conclude that $1 + 2i$ and $-1 - 2i$ are the two square roots of $-3 + 4i$.

Solutions to Exercises 1.2

1. Note that $z = 1-i$, $-z = -(1-i) = -1+i$ and $\bar{z} = \overline{1-i} = 1+i$. Thus z has coordinates $(1,-1)$, $-z$ has coordinates $(-1,1)$ and $\bar{z}$ has coordinates $(1,1)$. Also $|z| = \sqrt{1^2+1^2} = \sqrt{2}$.



5. Note that $z = \overline{1-i} = 1+i$. Hence $-z = -(1+i) = -1-i$ and $\overline{z} = \overline{1+i} = 1-i$. Thus z has coordinates $(1,1)$, $-z$ has coordinates $(-1,-1)$ and $\overline{z}$ has coordinates $(1,-1)$. Also $|z| = \sqrt{1^2+1^2} = \sqrt{2}$.



9. We use the property that $|ab| = |a||b|$ twice below:

$$|(1+i)(1-i)(1+3i)| = |1+i||(1-i)(1+3i)| = \overbrace{|1+i|}^{\sqrt{2}}\overbrace{|1-i|}^{\sqrt{2}}\overbrace{|1+3i|}^{\sqrt{10}} = 2\sqrt{10}.$$

13. Using the properties $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $|\overline{z}| = |z|$ we have

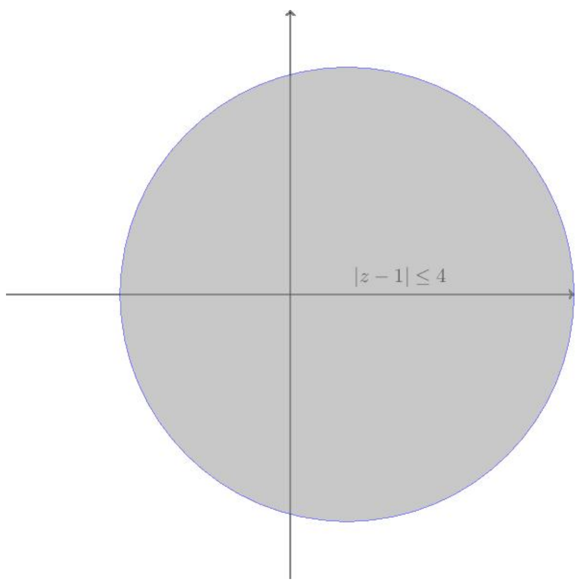
$$\left|\frac{i}{\overline{2-i}}\right| = \frac{|i|}{|\overline{2-i}|} = \frac{|i|}{|2-i|} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

17. The equation $|z - i| = -1$ does not have any solution because $|z|$ is a distance from the point z to the origin. And clearly a distance is always non-negative.

21. Let $z = x + iy$ and we have

$$\begin{aligned} |z-1| &\le 4 \\ |x+iy-1| &\le 4 \\ |(x-1)+iy|^2 &\le 4^2 \qquad \text{(Squaring both sides)} \\ (x-1)^2+y^2 &\le 16 \end{aligned}$$

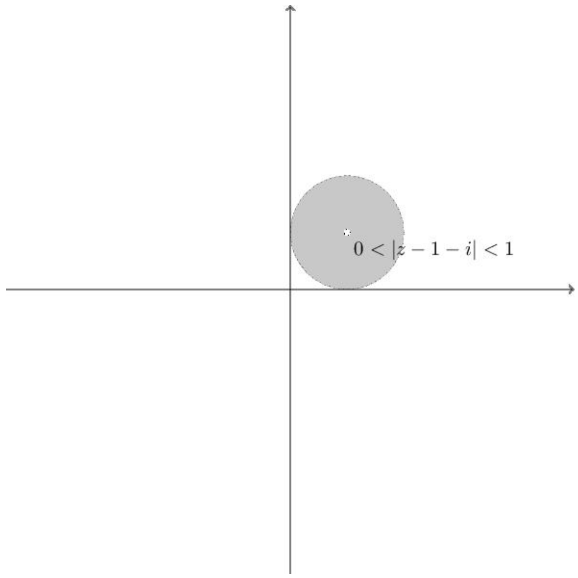
The inequality $|z - 1| \le 4$ represents a closed disc with radius 4 units and centre at $z = 1$ as shown below.



25. Let $z = x + iy$ and we have

$$\begin{aligned} 0 < |z-1-i| &< 1 \\ 0 < |x+iy-1-i| &< 1 \\ 0 < |(x-1)+i(y-1)|^2 &< 1^2 \\ 0 < (x-1)^2+(y-1)^2 &< 1 \end{aligned}$$

The inequality $0 < |z - 1 - i| < 1$ represents a puntured open disc with radius 1 units, centre at $z = 1 + i$ and puctured at $z = 1 + i$ as shown below.



29. (a) Let $z = x + iy$ and we have

$$\begin{aligned} \mathrm{Re}(z) &= a \\ \mathrm{Re}(x+iy) &= a \\ x &= a \end{aligned}$$

Thus the equation $\mathrm{Re}(z) = a$ represents the vertical line $x = a$.
(b) Let $z = x + iy$ and we have

$$\begin{aligned} \mathrm{Im}(z) &= b \\ \mathrm{Im}(x+iy) &= b \\ y &= b \end{aligned}$$

Thus the equation $\mathrm{Im}(z) = b$ represents the horizontal line $y = b$.
(c) Let $z = x + iy$ and let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be distinct points. We have

$$\begin{aligned} z &= z_1 + t(z_2 - z_1) \qquad (t \text{ is a real variable}) \\ x + iy &= (x_1 + iy_1) + t(x_2 + iy_2 - x_1 - iy_1) \\ x + iy &= \{x_1 + t(x_2 - x_1)\} + i\{y_1 + t(y_2 - y_1)\} \end{aligned}$$

By comparing the real and imaginary parts we get

$$\begin{aligned} x &= x_1 + t(x_2 - x_1) \\ y &= y_1 + t(y_2 - y_1) \end{aligned}$$

From the first equation we have

$$\begin{aligned} x - x_1 &= t(x_2 - x_1) \\ \frac{x - x_1}{x_2 - x_1} &= t \end{aligned}$$

From the second equation we have

$$\begin{aligned} y - y_1 &= t(y_2 - y_1) \\ \frac{y - y_1}{y_2 - y_1} &= t \end{aligned}$$

Now by eliminating t from both the equations, we get

$$\begin{aligned} \frac{x - x_1}{x_2 - x_1} &= \frac{y - y_1}{y_2 - y_1} \\ \frac{x - x_1}{y - y_1} &= \frac{x_2 - x_1}{y_2 - y_1} \end{aligned}$$

which represents the equation of a line passing through distinct points (x_1, y_1) and (x_2, y_2).

33. We know that

$$\begin{aligned} z.\overline{z} &= |z|^2 \\ z.\left(\frac{\overline{z}}{|z|^2}\right) &= 1 \qquad \text{(Dividing both sides by } |z|^2\text{)} \end{aligned}$$

Thus we get that $z^{-1} = \dfrac{\overline{z}}{|z|^2}$.

37. (a) We get the estimate

$$|\cos\theta + i\,\sin\theta| \leq |\cos\theta| + |i\,\sin\theta|$$

by the triangle inequality. Now we notice that

$$|i\,sin\theta| = |i|\,|\sin\theta| = 1|\sin\theta| = |\sin\theta|.$$

So,

$$|\cos\theta| + |i\,\sin\theta| = |\cos\theta| + |\sin\theta|.$$

We know from algebra that $|\sin\theta| \leq 1$, and $|\cos\theta| \leq 1$ Therefore we have

$$|\cos\theta| + |\sin\theta| \leq 1 + 1 = 2,$$

and this justifies the last step of the estimation above.
(b) By definition of the absolute value we know that if $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$. Therefore if $z = \cos\theta + i\,\sin\theta$ then $|z| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2}$. But from trigonometry we know the formula

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

is true for any number θ. Therefore we have

$$|\cos\theta + i\,\sin\theta| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1.$$

41. We want to know something about $|\frac{1}{z-4}|$ given some information about $|z-1|$. Notice that an upper bound of the quantity $|\frac{1}{z-4}|$ is given by the reciprocal of any lower bound for $|z-4|$. So, first we can find some information about $|z-4|$. For this we use some ideas from the example 8 in this section. We notice that $z-4=(z-1)-3$. Therefore by the inequality $|z_1-z_2| \geq ||z_1|-|z_2||$ we have

$$|z-4| = |(z-1)-3| \geq ||z-1|-3|$$

Now since we know $|z-1| \leq 1$ we get that $||z-1|-3| \geq |1-3| = 2$. Hence $|z-4| \geq 2$. Finally taking reciprocals of the sides of the inequality we will reverse the sign of the inequality and get the needed upper estimate:

$$\frac{1}{|z-4|} \leq \frac{1}{2}.$$

45. (a) By triangle inequality we have

$$\left|\sum_{j=1}^{n} \overline{v_j} w_j\right| \leq \sum_{j=1}^{n} |\overline{v_j} w_j|.$$

And now we use the properties $|z_1 z_2| = |z_1||z_2|$ and $|\overline{z}| = |z|$ and get

$$|\overline{v_j} w_j| = |\overline{v_j}||w_j| = |v_j||w_j|.$$

Using this identity in the triangle inequality above and recalling the assumption that we already proved () we have

$$\begin{aligned} \left|\sum_{j=1}^{n} \overline{v_j} w_j\right| &\leq \sum_{j=1}^{n} |\overline{v_j} w_j| \\ = \sum_{j=1}^{n} |v_j||w_j| &\leq \sqrt{\sum_{j=1}^{n} |v_j|^2}\sqrt{\sum_{j=1}^{n} |w_j|^2}. \end{aligned}$$

(b) Using the hint we start from the obvious inequality

$$0 \leq \sum_{j=1}^{n} \Big(|v_j| - |w_j|\Big)^2.$$

Expanding the right hand side of this inequality we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} \Big(|v_j| - |w_j|\Big)^2 = \sum_{j=1}^{n} \Big(|v_j|^2 - 2|v_j||w_j| + |w_j|^2\Big) \\ &= \sum_{j=1}^{n} \Big(|v_j|^2 + |w_j|^2\Big) - 2\sum_{j=1}^{n} |v_j||w_j|. \end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=1}^{n}|v_j||w_j| &\le \frac{1}{2}\sum_{j=1}^{n}\Big(|v_j|^2+|w_j|^2\Big)=\frac{1}{2}\Big(\sum_{j=1}^{n}|v_j|^2+\sum_{j=1}^{n}|w_j|^2\Big)\\ &= \frac{1}{2}(1+1)=1=\sqrt{1}\cdot\sqrt{1}=\sqrt{\sum_{j=1}^{n}|v_j|^2}\sqrt{\sum_{j=1}^{n}|w_j|^2}.\end{aligned}$$

(c) Using the hint we can consider $v=(v_1,\, v_2,\, \dots,\, v_n)$ and $w=(w_1,\, w_2,\, \dots,\, w_n)$ and look at them as at vectors. If we define $\|v\|=\sqrt{\sum_{j=1}^{n}|v_j|^2}$ and $\|w\|=\sqrt{\sum_{j=1}^{n}|w_j|^2}$. Then it turns out we need to prove the following inequality:

$$\sum_{j=1}^{n}|v_j|\,|w_j|\le \|v\|\|w\|.$$

In order to do so we define new vectors $U=\frac{1}{\|u\|}u$ and $W=\frac{1}{\|w\|}w$. So, the coordinates of these vectors are correspondingly $U_j=\frac{u_j}{\|u\|}$ and $W_j=\frac{w_j}{\|w\|}$. Then we have

$$\sum_{j=1}^{n}|U_j|^2=\sum_{j=1}^{n}\left|\frac{u_j}{\|u\|}\right|^2=\frac{1}{\|u\|^2}\sum_{j=1}^{n}|u_j|^2=\frac{1}{\|u\|^2}\|u\|^2=1,$$

and similar we get $\sum_{j=1}^{n}|W_j|^2=1$. Therefore we can apply part (b) to the vectors V and W. We have

$$\sum_{j=1}^{n}|V_j|\,|W_j|\le\sqrt{\sum_{j=1}^{n}|V_j|^2}\sqrt{\sum_{j=1}^{n}|W_j|^2}.$$

The left side is equal to

$$\sum_{j=1}^{n}\frac{|v_j|}{\|v\|}\,\frac{|w_j|}{\|w\|}=\frac{1}{\|v\|\|w\|}\sum_{j=1}^{n}|v_j||w_j|.$$

The right side is equal to

$$\begin{aligned}\sqrt{\sum_{j=1}^{n}\frac{|v_j|^2}{\|v\|^2}}\sqrt{\sum_{j=1}^{n}\frac{|w_j|^2}{\|w\|^2}} &= \sqrt{\frac{1}{\|v\|^2}\sum_{j=1}^{n}|v_j|^2}\sqrt{\frac{1}{\|w\|^2}\sum_{j=1}^{n}|w_j|^2}\\ &= \frac{1}{\|v\|\|w\|}\sqrt{\sum_{j=1}^{n}|v_j|^2}\sqrt{\sum_{j=1}^{n}|w_j|^2}\\ &= \frac{1}{\|v\|\|w\|}\|u\|\|v\|=1.\end{aligned}$$

Hence we can rewrite the inequality we received in the form:

$$\frac{1}{\|v\|\|w\|}\sum_{j=1}^{n}|v_j||w_j|\le 1.$$

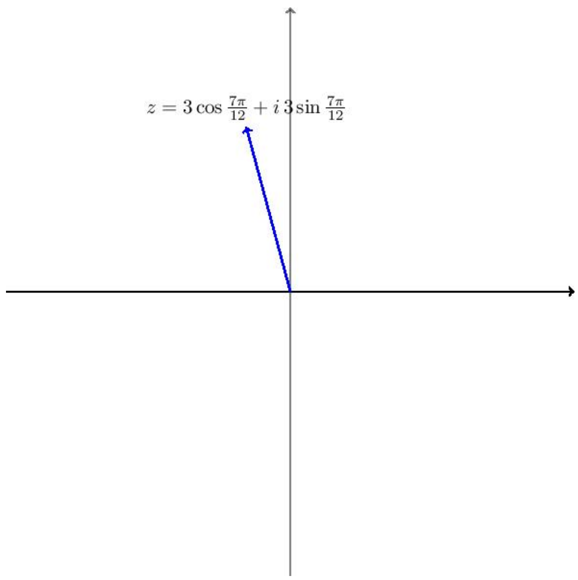
So, if we multiply both sides by $\|v\|\|w\|$ we get the needed inequality.

Solutions to Exercises 1.3

1. We need to present the number given in its polar form in the form with the real and imaginary parts $z = x + iy$. We have

$$z = 3\left(\cos\frac{7\pi}{12} + i\,\sin\frac{7\pi}{12}\right) = 3\cos\frac{7\pi}{12} + i\,3\sin\frac{7\pi}{12}$$

In Cartesian coordinates z is represented by $(3\cos\frac{7\pi}{12}, 3\sin\frac{7\pi}{12})$ as shown below.



5. Let $z = -3 - 3\,i$. Then we have $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{9 \cdot 2} = 3\sqrt{2}$. Also, we can find the argument by evaluating

$$\cos\theta = \frac{x}{r} = \frac{-3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\theta = \frac{y}{r} = \frac{-3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

From the Table 1 in the Section 1.3 we see that $\theta = \frac{5\pi}{4}$. Thus, $\arg z = \frac{5\pi}{4} + 2k\pi$. Since $\frac{5\pi}{4}$ is not from the interval $(-\pi, \pi]$ we can subtract 2π and get that $\operatorname{Arg} z = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$. So, the polar representation is

$$-3 - 3\,i = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\,\sin\left(-\frac{3\pi}{4}\right)\right).$$

9. Again we can denote $z = -\frac{i}{2}$. Then we have $r = \sqrt{(-\frac{1}{2})^2} = \frac{1}{2}$. We can evaluate

$$\cos\theta = \frac{x}{r} = \frac{0}{1/2} = 0 \quad \text{and} \quad \sin\theta = \frac{y}{r} = \frac{-1/2}{1/2} = -1.$$

And, from the Table 1 in the Section 1.3 we find $\theta = \frac{3\pi}{2}$. (Also we could plot the complex number $(-\frac{i}{2})$ as a point in the complex plane and find the angle from the picture.) Therefore

$\arg z = \frac{3\pi}{2} + 2\pi k$. Since $\frac{3\pi}{2}$ is not from the interval $(-\pi, \pi]$ we can subtract 2π and get that $\text{Arg}\ \ z = \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$. So, the polar representation is

$$-\frac{i}{2} = \frac{1}{2}\left(\cos\left(-\frac{\pi}{2}\right) + i\,\sin\left(-\frac{\pi}{2}\right)\right).$$

13. We have $z = x + i\,y = 13 + i\,2$. Since $x > 0$ we compute

$$\text{Arg}\, z = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{2}{13}) \approx 0.153.$$

Hence we express $\arg z \approx 0.153 + 2\pi k$ for all integer k.

17. First, we need to express the number $z = -\sqrt{3} + i$ in the polar form. We compute $r = \sqrt{3+1} = 2$. And since $-\sqrt{3} < 0$, and $1 > 0$ we see that $Arg\ z = tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{5\pi}{6}$. Since we need to find the cube of the number z we can use the De Moivre's Identity to get the polar representation:

$$\begin{aligned}
(-\sqrt{3}+i)^3 &= \left(2\left(\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right)\right)^3 \\
&= 2^3\left(\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right)^3 \\
&= 8\left(\cos\left(\frac{15\pi}{6}\right) + i\,\sin\left(\frac{15\pi}{6}\right)\right) \\
&= 8\left(\cos\left(\frac{3\pi}{6}\right) + i\,\sin\left(\frac{3\pi}{6}\right)\right).
\end{aligned}$$

Where in the last identity we used the fact that $Arg\ (-\sqrt{3}+i)^3 = \frac{15\pi}{6} - 2\pi = \frac{3\pi}{6}$ should be in the interval $(-\pi, \pi]$.

21. First, we find the modulus and the argument of the number $z = 1 + i$. We have $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. And, since for $z = x + iy = 1 + i1$ we get $x > 0$ we compute $\text{Arg}\ \ z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$. Hence z can be expressed in the polar form

$$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right).$$

After this we can use De Moivre's identity to get

$$\begin{aligned}
(1+i)^{30} &= \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right)\right)^{30} \\
&= (\sqrt{2})^{30}\left(\cos\left(\frac{30\pi}{4}\right) + i\,\sin\left(\frac{30\pi}{4}\right)\right) \\
&= 2^{15}\,(0 + i\cdot 1) = 2^{15}\,i.
\end{aligned}$$

Thus $\text{Re}\ \left((1+i)^{30}\right) = 0$, and $\text{Im}\ \left((1+i)^{30}\right) = 2^{15}$.

25. (a) Set $z_1 = -1 = z_2$. Then $\text{Arg}(z_1 z_2) = \text{Arg}(1) = 0$, whereas $\text{Arg}(z_1) + \text{Arg}(z_2) = \pi + \pi = 2\pi$. Thus $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$.
(b) Set $z_1 = 0$ and $z_2 = -1$. Then $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = \text{Arg}(0) = 0$, whereas $\text{Arg}(z_1) - \text{Arg}(z_2) = 0 - \pi = -\pi$. Thus $\text{Arg}\left(\dfrac{z_1}{z_2}\right) \neq \text{Arg}(z_1) - \text{Arg}(z_2)$.
(c) Set $z = -1$. Then $\text{Arg}(\bar{z}) = \text{Arg}(1) = 0$, whereas $-\text{Arg}(z) = -\text{Arg}(-1) = -\pi$. Thus $\text{Arg}(\bar{z}) \neq -\text{Arg}(z)$.
(d) Set $z = -1$. Then $\text{Arg}(-z) = \text{Arg}(1) = 0$, whereas $\text{Arg}(z) + \pi = \text{Arg}(-1) + \pi = \pi + \pi = 2\pi$. Thus $\text{Arg}(-z) \neq \text{Arg}(z) + \pi$.

29. We know that for any complex numbers $z_1 = r_1(\cos\theta_1 + i\,\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\,\sin\theta_2)$ we have

$$\begin{aligned} z_1\, z_2 &= r_1[\cos(\theta_1) + i\,\sin(\theta_1)] r_2[\cos(\theta_2) + i\,\sin(\theta_2)] \\ &= r_1\, r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

We can use this property several times to get the identity above step by step.

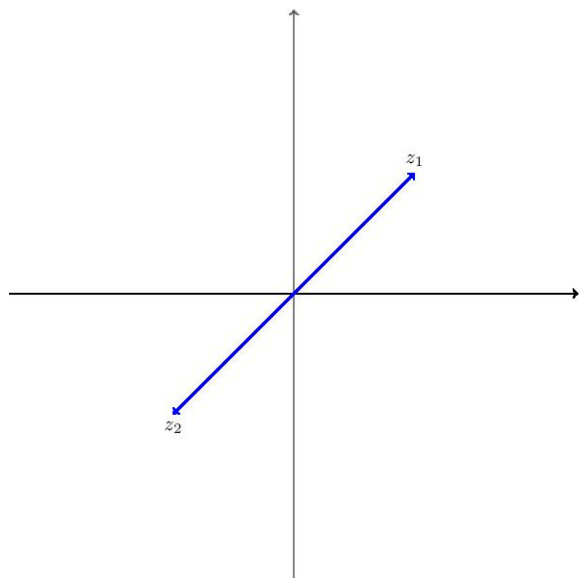
$$\begin{aligned} z_1 z_2 \cdots z_n &= (z_1\, z_2) z_3 \cdots z_n \\ &= (r_1(\cos(\theta_1) + i\,\sin(\theta_1))\, r_2(\cos(\theta_2) + i\,\sin(\theta_2))) z_3 \cdots z_n \\ &= (r_1\, r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) z_3 z_4 \cdots z_n \\ &= (r_1\, r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) r_3(\cos(\theta_3) + i\,\sin(\theta_3)) z_4 \cdots z_n \\ &= (r_1\, r_2\, r_3)(\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)) z_4 z_5 \cdots z_n \\ &= \cdots \\ &= (r_1\, r_2 \cdots r_{n-1})(\cos(\theta_1 + \theta_2 + \cdots + \theta_{n-1}) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_{n-1})) z_n \\ &= (r_1\, r_2 \cdots r_{n-1})(\cos(\theta_1 + \theta_2 + \cdots + \theta_{n-1}) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_{n-1})) \\ &\qquad \cdot r_n(\cos(\theta_n) + i\,\sin(\theta_n)) \\ &= (r_1\, r_2 \cdots r_{n-1}\, r_n) \\ &\qquad \cdot(\cos(\theta_1 + \theta_2 + \cdots + \theta_{n-1} + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_{n-1} + \theta_n)). \end{aligned}$$

33. We have

$$\begin{aligned} z^2 &= i \\ z^2 &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \end{aligned}$$

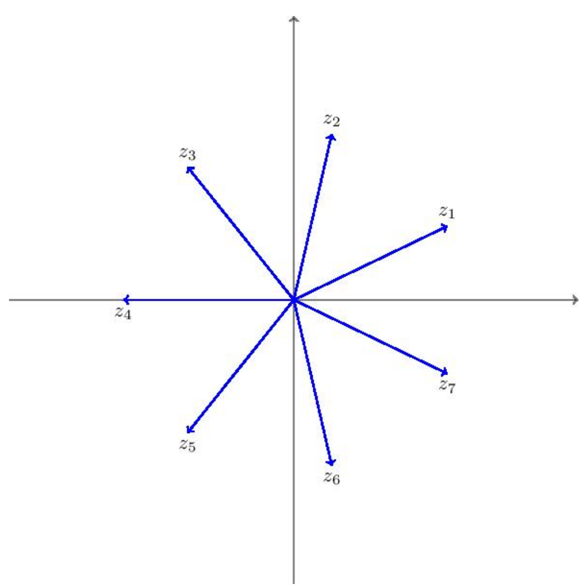
So by the formula for the n-th roots with $n = 2$, we find the roots to be

$$\begin{aligned} z_1 &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \\ z_2 &= \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} \end{aligned}$$



37. In order to solve this equation we need to find the 7-th root of (-7). Thus we express (-7) in the polar form. $-7 = 7(\cos(\pi) + i\,\sin(\pi))$. Hence by the formula for the n-th root with $n = 7$ we have that the solutions are

$$\begin{aligned}
z_1 &= \sqrt[7]{7}\,(\cos(\pi/7) + i\,\sin(\pi/7))\,,\ z_2 = \sqrt[7]{7}\,(\cos(3\pi/7) + i\,\sin(3\pi/7))\,,\\
z_3 &= \sqrt[7]{7}\,(\cos(5\pi/7) + i\,\sin(5\pi/7))\,,\ z_4 = \sqrt[7]{7}\,(\cos(7\pi/7) + i\,\sin(7\pi/7))\,,\\
z_5 &= \sqrt[7]{7}\,(\cos(9\pi/7) + i\,\sin(9\pi/7))\,,\ z_6 = \sqrt[7]{7}\,(\cos(11\pi/7) + i\,\sin(11\pi/7))\,,\quad \text{and}\\
z_7 &= \sqrt[7]{7}\,(\cos(13\pi/7) + i\,\sin(13\pi/7))\,.
\end{aligned}$$

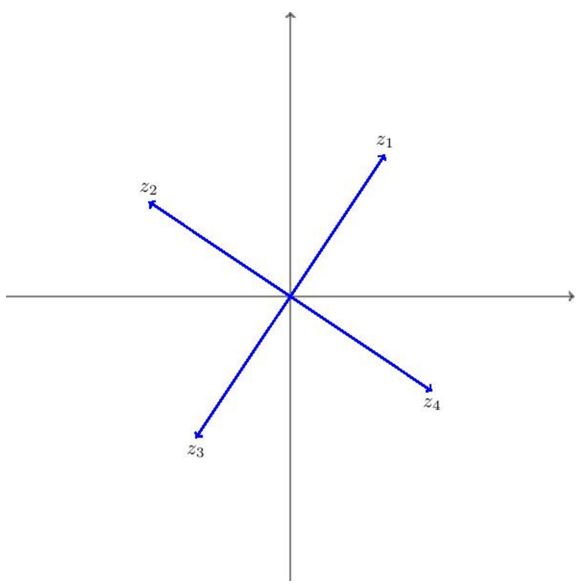


41. We have

$$\begin{aligned} z^4 &= -1 - i \\ z^4 &= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right). \end{aligned}$$

So by the formula for the n-th roots with $n = 4$, we find the roots to be

$$\begin{aligned} z_1 &= \sqrt[8]{2}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right) \\ z_2 &= \sqrt[8]{2}\left(\cos\frac{13\pi}{16} + i\sin\frac{13\pi}{16}\right) \\ z_3 &= \sqrt[8]{2}\left(\cos\frac{21\pi}{16} + i\sin\frac{21\pi}{16}\right) \\ z_4 &= \sqrt[8]{2}\left(\cos\frac{29\pi}{16} + i\sin\frac{29\pi}{16}\right). \end{aligned}$$



45. We have

$$\begin{aligned} (z+2)^3 &= 3i \\ w^3 &= 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \qquad \text{(Set } w = z+2\text{)} \end{aligned}$$

So by the formula for n-th roots for $n = 3$, we find the roots to be

$$\begin{aligned}
w_1 &= \sqrt[3]{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\\
&= \sqrt[3]{3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\\
&= \frac{(\sqrt[6]{3})^5}{2} + i\frac{\sqrt[3]{3}}{2}\\
w_2 &= \sqrt[3]{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\\
&= \sqrt[3]{3}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\\
&= -\frac{(\sqrt[6]{3})^5}{2} + i\frac{\sqrt[3]{3}}{2}\\
w_3 &= \sqrt[3]{3}\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right)\\
&= -\sqrt[3]{3}i
\end{aligned}$$

Since $z = w - 2$, the solutions of the original problem are

$$\begin{aligned}
z_1 &= w_1 - 2\\
&= \left(\frac{(\sqrt[6]{3})^5}{2} - 2\right) + i\frac{\sqrt[3]{3}}{2}\\
z_2 &= w_2 - 2\\
&= -\left(\frac{(\sqrt[6]{3})^5}{2} + 2\right) + i\frac{\sqrt[3]{3}}{2}\\
z_3 &= w_3 - 2\\
&= -2 - \sqrt[3]{3}i.
\end{aligned}$$

49. We have

$$\begin{aligned}
z^2 + z + 1 - i &= 0\\
z &= \frac{-1 \pm \sqrt{1^2 - 4(1-i)}}{2} \qquad \text{(Using the quadratic formula)}\\
&= \frac{-1 \pm \sqrt{-3+4i}}{2}
\end{aligned}$$

Now let

$$\begin{aligned}
w^2 &= -3 + 4i\\
w^2 &= 5\left(-\frac{3}{5} + i\frac{4}{5}\right)\\
w^2 &= 5(\cos\theta + \sin\theta) \qquad \left(\text{where } \frac{\pi}{2} < \theta < \pi \text{ and } \cos\theta = -\frac{3}{5},\ \sin\theta = \frac{4}{5}\right)
\end{aligned}$$

So the principal square root is

$$w = \sqrt{5}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

Now we use the half-angle identity i.e. $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$, $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ and compute

$$\begin{aligned} w &= \sqrt{5}\left(\sqrt{\frac{1-\frac{3}{5}}{2}} + i\sqrt{\frac{1+\frac{3}{5}}{2}}\right) \\ &= 1+2i. \end{aligned}$$

Thus the original solutions are

$$\begin{aligned} z_1 &= \frac{-1+w}{2} = i \\ z_2 &= \frac{-1-w}{2} = -1-i. \end{aligned}$$

53. We have

$$\begin{aligned} z^4 - (1+i)z^2 + i &= 0 \\ u^2 - (1+i)u + i &= 0 \qquad \text{(Set } u = z^2\text{)} \\ u &= \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4i}}{2} \qquad \text{(Using the quadratic formula)} \\ u &= \frac{-(1+i) \pm \sqrt{-2i}}{2} \\ u &= \frac{-(1+i) \pm (-1+i)}{2} \qquad \text{(The principal square root of } -2i \text{ is } -1+i\text{, see problem 51)} \\ u &= -1, -i. \end{aligned}$$

Since $z = \sqrt{u}$, we get

$$\begin{aligned}
z_1 &= \sqrt{-1} \\
&= i \\
z_2 &= -\sqrt{-1} \\
&= -i \\
z_3 &= \sqrt{-i} \\
&= \sqrt{\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}} \\
&= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \\
&= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\
z_4 &= -\sqrt{-i} \\
&= -z_3 \\
&= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.
\end{aligned}$$

57. By De Moivre's identity for $n = 3$ we have

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

Now by comparing the real and imaginary parts we get

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta.
\end{aligned}$$

61. We have $z^n = 1$ and by the formula for n-th roots we get

$$\begin{aligned}
z^n &= \cos 0 + i\sin 0 \\
\omega_k &= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \qquad \text{(where } k = 0, \cdots, n-1\text{)}.
\end{aligned}$$

65. We prove the binomial formula for complex numbers by induction on n. The statement for $n = 1$ can be easily verified. By induction hypothesis, we assume that the statement is true for n and we prove it for $n + 1$.

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n.(a+b) \\
&= \left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}.b^m\right)(a+b)
\end{aligned}$$

By expanding the product we see that the coefficient of $a^{n+1-i}b^i$ in $(a+b)^{n+1}$ is equal to the sum of the coefficient of $a^{n-i}b^i$ in $(a+b)^n$ i.e. $\binom{n}{i}$ and the coefficient of $a^{n+1-i}b^{i-1}$ in $(a+b)^n$ i.e. $\binom{n}{i-1}$. Thus we get

$$\begin{aligned}
(a+b)^{n+1} &= a^{n+1} + \left(\sum_{m=1}^{n} \left(\binom{n}{m} + \binom{n}{m-1}\right) a^{n+1-m} b^m\right) + b^{n+1} \\
&= a^{n+1} + \left(\sum_{m=1}^{n} \binom{n+1}{m} a^{n+1-m} b^m\right) + b^{n+1} \qquad \left(\text{use } \binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}\right) \\
&= \sum_{m=0}^{n+1} \binom{n+1}{m} a^{n+1-m} b^m \qquad \left(\text{By convention } \binom{n+1}{0} = 1 = \binom{n+1}{n+1}\right).
\end{aligned}$$

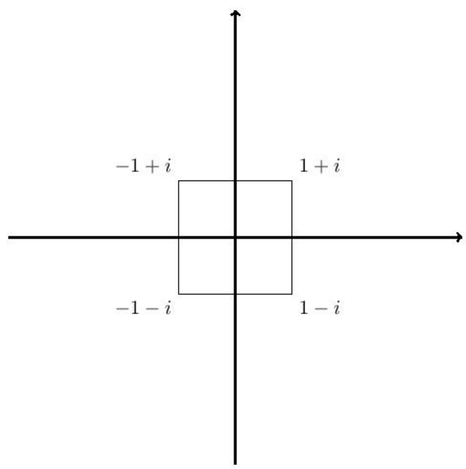
We have thus proved the statement for $n+1$ which shows that the binomial theorem holds true for complex numbers.

Solutions to Exercises 1.4

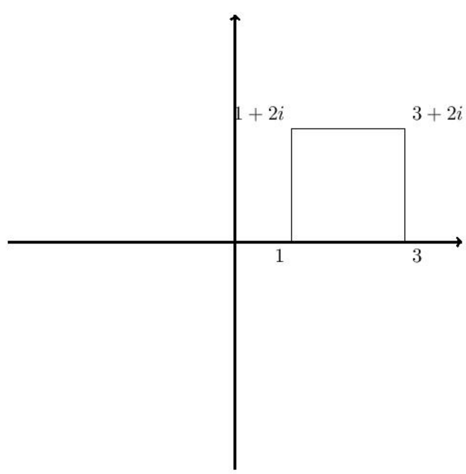
1. We are given $f(z) = iz + 2 + i$. Now

$$\begin{aligned} f(1+i) &= i(1+i) + 2 + i = 1 + 2i \\ f(-1+i) &= i(-1+i) + 2 + i = 1 \\ f(-1-i) &= i(-1-i) + 2 + i = 3 \\ f(1-i) &= i(1-i) + 2 + i = 3 + 2i. \end{aligned}$$

5. Let S be the square with vertices $1+i, -1+i, -1-i, 1-i$ which is shown below.



Then $f[S]$ is also a square with vertices $1, 3, 3+2i, 1+2i$ which is shown below.



9. Observe that

$$f(z) = (-1-i)z + 3 + i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) z + 3 + i = \sqrt{2}e^{-\frac{3i\pi}{4}} z + (3+i)$$

Thus $f(z)$ is obtained by rotating z clockwise by $\dfrac{3\pi}{4}$, then stretching by a factor of $\sqrt{2}$ and then translating 1 unit up and 3 units to the right.

13. We are given $f(z) = z^2 - 2z + i$. Now set $z = x + iy$ and we get

$$f(x+iy) = (x+iy)^2 - 2(x+iy) + i = x^2 + 2ixy - y^2 - 2x - 2iy + i = (x^2 - y^2 - 2x) + i(2xy - 2y + 1)$$

Therefore $u(x, y) = x^2 - y^2 - 2x$ and $v(x, y) = 2xy - 2y + 1$.

17. We are given $f(z) = 3\operatorname{Arg}(z)$. Now set $z = x + iy$ and we get

$$f(x + iy) = 3\operatorname{Arg}(x + iy)$$

Therefore

$$u(x, y) = \begin{cases} 3\tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \\ 3\tan^{-1}\left(\frac{y}{x}\right) + 3\pi & \text{if } x < 0, y \geq 0 \\ 3\tan^{-1}\left(\frac{y}{x}\right) - 3\pi & \text{if } x < 0, y < 0 \end{cases}$$

and $v(x, y) = 0$.

21. Let $f(z) = az + b$ and we have

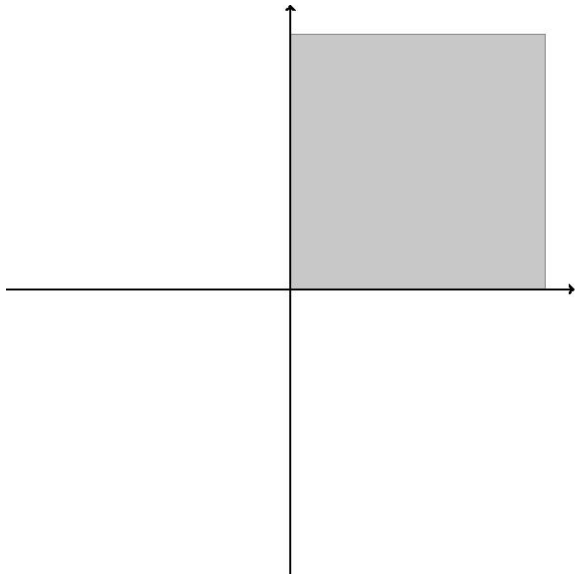
$$\begin{aligned} f(1) &= 3 + i \\ a + b &= 3 + i \\ a &= 3 + i - b \end{aligned}$$

Also we have

$$\begin{aligned} f(3i) &= -2 + 6i \\ 3ia + b &= -2 + 6i \\ 3i(3 + i - b) + b &= -2 + 6i \qquad \text{(Substituting the expression for } a\text{)} \\ -3 + 9i + (1 - 3i)b &= -2 + 6i \\ (1 - 3i)b &= 1 - 3i \\ b &= 1 \end{aligned}$$

So $a = 3 + i - 1 = 2 + i$. Thus $f(z) = (2 + i)z + 1$.

25. We are given $S = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\}$ as shown by the shaded region below.



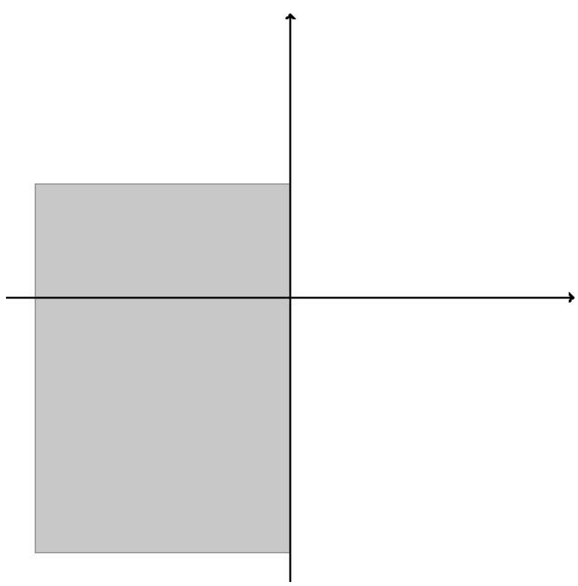
An arbitrary point of S is of the form $z = x + iy$, where $x > 0$ and $y > 0$. Then under the mapping $f(z) = -z + 2i$, we have

$$f(z) = -(x + iy) + 2i = -x + i(2 - y)$$

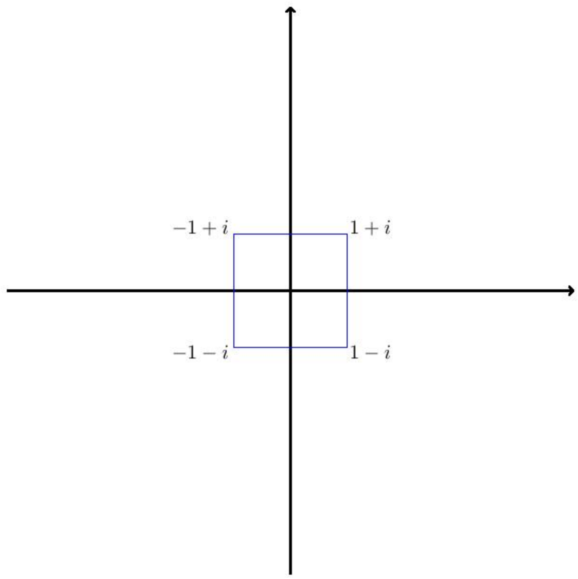
We hence note that $\text{Re}(f(z))$ ranges in the interval $-\infty < \text{Re}(f(z)) < 0$ and $\text{Im}(f(z))$ ranges in the interval $-\infty < \text{Re}(f(z)) < 2$. So

$$f[S] = \{z \in \mathbb{C} : \text{Re}(z) < 0, \text{Im}(z) < 2\}$$

as shown by the shaded region below.



29. S is a square with vertices $1 + i, -1 + i, -1 - i, 1 - i$ which is shown below.



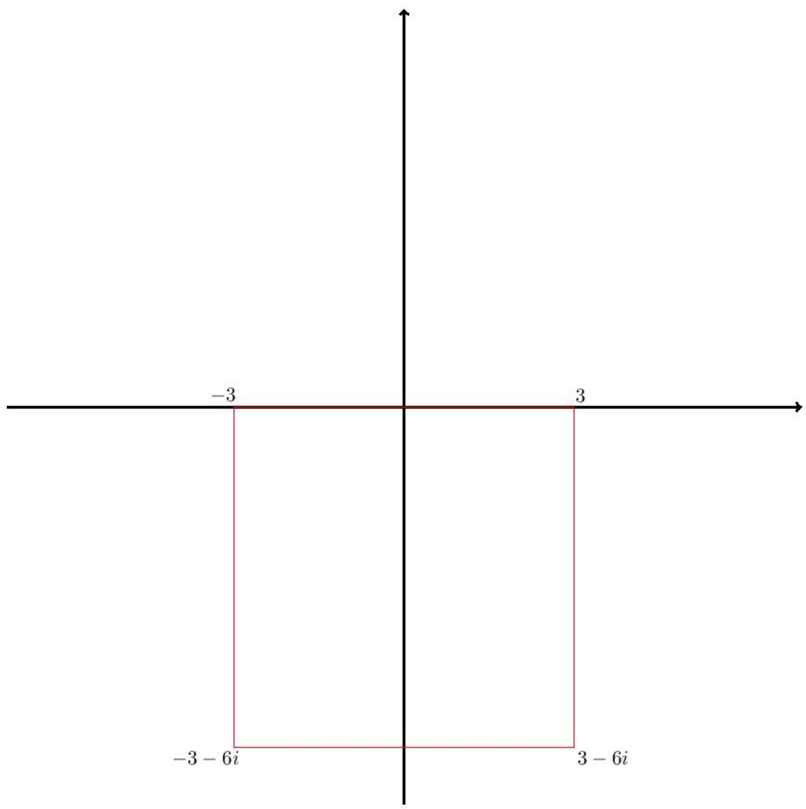
We see that the transformation f is given by

$$f(z) = 3(z+1)e^{-\frac{i\pi}{2}} = -3(z+1)i$$

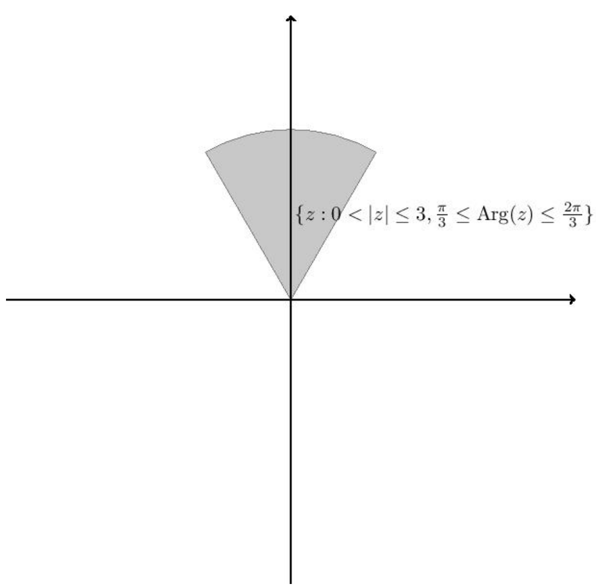
Also note that

$$\begin{aligned} f(1+i) &= 3-6i \\ f(-1+i) &= 3 \\ f(-1-i) &= -3 \\ f(1-i) &= -3-6i \end{aligned}$$

$f[S]$ is a square with vertices given by $(3,-6), (3,0), (-3,0), (-3,-6)$ which is shown below.



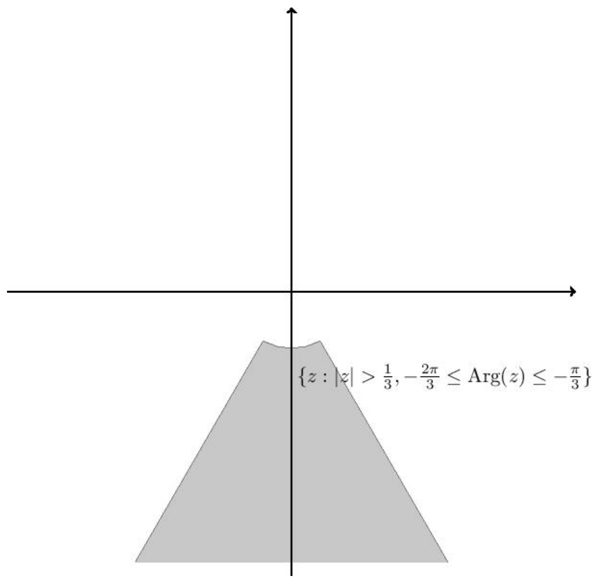
33. We are given $S = \{z \in \mathbb{C} : 0 < |z| \le 3, \dfrac{\pi}{3} \le \operatorname{Arg}(z) \le \dfrac{2\pi}{3}\}$ as shown by the shaded region below.



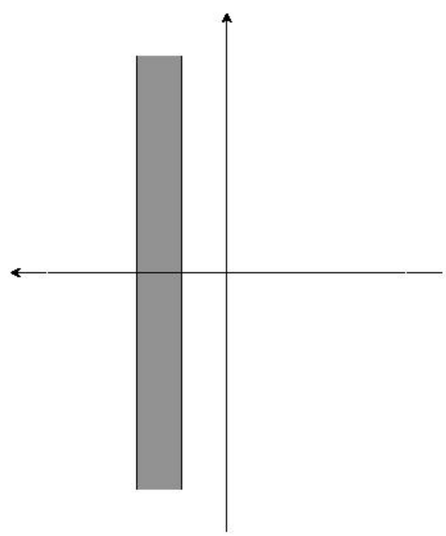
If we write $z = r(\cos\theta + i\,\sin\theta)$, then $f(z) = \frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\,\sin(-\theta))$. Hence the polar coordinates of $w = f(z) = \rho(\cos\phi + i\sin\phi)$ are $\frac{1}{3} < \rho < \infty$, and $-\frac{2\pi}{3} \le Arg\ w \le -\frac{\pi}{3}$. As r increases from 0 to 3, ρ decreases from ∞ to $\frac{1}{3}$; and as θ goes from $\frac{\pi}{3}$ up to $\frac{2\pi}{3}$, ϕ decreases from $(-\frac{\pi}{3})$ to $(-\frac{2\pi}{3})$. Thus

$$f[S] = \{z \in \mathbb{C} : \frac{1}{3} < |z| < \infty, -\frac{2\pi}{3} \le \text{Arg}(z) \le -\frac{\pi}{3}\}$$

as shown by the shaded region below.



37. We are given $S = \{x + iy : -2 \le x \le 0\}$ and is represented by the shaded region below.



Now consider a vertical strip $L_{x_0} = \{x + iy : x = x_0\}$ of S. Let z be an arbitrary point of L_{x_0} and assume $x_0 > 0$. Then under the mapping $f(z) = z^2$, we have

$$f(z) = (x_0 + iy)^2 = (x_0^2 - y^2) + i(2x_0y)$$

We set $u = \mathrm{Re}(f(z)) = x_0^2 - y^2$ and $v = \mathrm{Im}(f(z)) = 2x_0 y$. By eliminating y, we get an algebraic relation between u and v, i.e.

$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

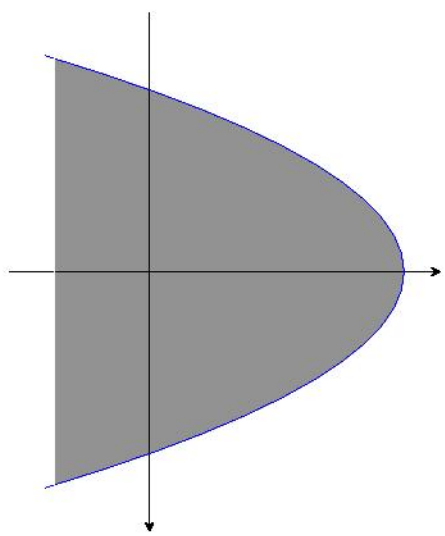
Thus we get

$$f[L_{x_0}] = \left\{u + iv : u = x_0^2 - \frac{v^2}{4x_0^2}\right\}$$

which is a leftward-facing parabola with vertex at $(x_0^2, 0)$ and v-intercepts at $(0, \pm 2x_0^2)$. Also it is not hard to see that $f[L_0] = (-\infty, 0]$. Since $S = \bigcup_{-2 \le x_0 \le 0} L_{x_0}$, we get

$$\begin{aligned} f[S] &= \bigcup_{-2 \le x_0 \le 0} f[L_{x_0}] \\ &= (-\infty, 0] \cup \bigcup_{-2 \le x_0 < 0} \left\{u + iv : u = x_0^2 - \frac{v^2}{4x_0^2}\right\} \\ &= \left\{u + iv : u \le 4 - \frac{v^2}{16}\right\}. \end{aligned}$$

In other words $f[S]$ is the region enclosed within the parabola $u = 4 - \dfrac{v^2}{16}$ as represented by the shaded region below.



41. We are given $S = \{x + iy : 0 \le y \le 2\}$. Now consider a horizontal strip $L_{y_0} = \{x + iy : y = y_0\}$ of S. Let z be an arbitrary point of L_{y_0} and assume $y_0 > 0$. Then under the mapping $f(z) = z^2$, we have

$$f(z) = (x + iy_0)^2 = (x^2 - y_0^2) + i(2xy_0)$$

We set $u = \mathrm{Re}(f(z)) = x^2 - y_0^2$ and $v = \mathrm{Im}(f(z)) = 2xy_0$. By eliminating x, we get an algebraic relation between u and v, i.e.

$$u = \frac{v^2}{4y_0^2} - y_0^2$$

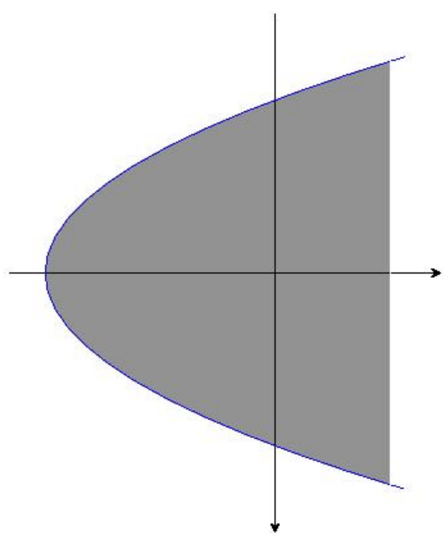
Thus we get

$$f[L_{y_0}] = \left\{u + iv : u = \frac{v^2}{4y_0^2} - y_0^2\right\}$$

which is a rightward-facing parabola with vertex at $(-y_0^2, 0)$ and v-intercepts at $(0, \pm 2y_0^2)$. Also it is not hard to see that $f[L_0] = [0, \infty)$. Since $S = \bigcup_{0 \le y_0 \le 2} L_{y_0}$, we get

$$\begin{aligned}
f[S] &= \bigcup_{0 \le y_0 \le 2} f[L_{y_0}] \\
&= [0, \infty) \cup \bigcup_{0 < y_0 \le 2} \left\{u + iv : u = \frac{v^2}{4y_0^2} - y_0^2\right\} \\
&= \left\{u + iv : u \ge \frac{v^2}{16} - 4\right\}
\end{aligned}$$

In other words $f[S]$ is the region enclosed within the parabola $u = \dfrac{v^2}{16} - 4$ as represented by the shaded region below.



45. We are given $S = \{z \in \mathbb{C} \setminus \{0\} : \frac{\pi}{4} \le \mathrm{Arg}(z) \le \frac{3\pi}{4}\} \cup \{0\}$. We consider the strip $L_{r_0} = \{z : |z| = r_0, \frac{\pi}{4} \le \mathrm{Arg}(z) \le \frac{3\pi}{4}\}$ of S. Let z be an arbitrary point of L_{r_0}, then under the mapping $f(z) = iz^2$ we get

$$f(z) = i\left(r_0(\cos\theta + i\sin\theta)\right)^2 = r_0^2\left(\cos\left(2\theta + \frac{\pi}{2}\right) + i\sin\left(2\theta + \frac{\pi}{2}\right)\right).$$

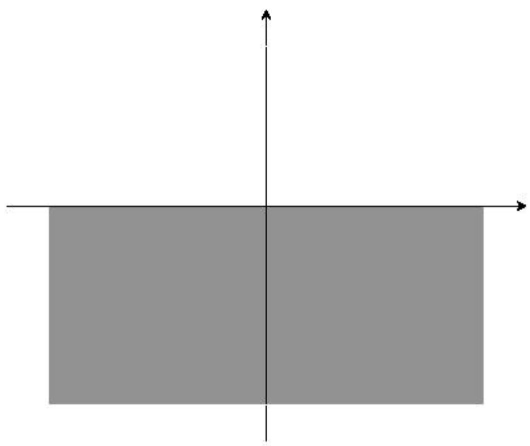
As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, $2\theta+\frac{\pi}{2}$ increases from π to 2π. Thus

$$f[L_{r_0}] = \{z \in \mathbb{C} : |z| = r_0^2, \pi \leq \operatorname{Arg}(z) \leq 2\pi\}$$

and $f(0)=0$. Since $S = \{0\} \cup \bigcup_{0<r_0<\infty} L_{r_0}$, we get

$$f[S] = \{f(0)\} \cup \bigcup_{0<r_0<\infty} f[L_{r_0}] = \{z \in \mathbb{C} \setminus \{0\} : \pi \leq \operatorname{Arg}(z) \leq 2\pi\} \cup \{0\}$$

In other words $f[S]$ is the lower half complex plane including the real axis as shown by the shaded region below.



49. (a) We compute

$$f(g(z)) = ag(z)+b = a(cz+d)+b = acz+ad+b = (ac)z+(ad+b).$$

This means that $f(g(z))$ is also linear.
(b) First, we express the number a in the polar form $a = r(\cos\theta + i\,\sin\theta)$. If we substitute this value to the function $f(z)$ we obtain

$$f(z) = r(\cos\theta + i\,\sin\theta)z + b = ((\cos\theta + i\,\sin\theta)(rz)) + b.$$

So, if we take $g_1(z) = z+b$, $g_2(z) = (\cos\theta + i\,\sin\theta)z$, and $g_3(z) = rz$ we can find a representation for $f(z)$:

$$f(z) \quad = \quad g_1((\cos\theta + i\,\sin\theta)(rz)) = g_1(g_2(rz)) = g_1(g_2(g_3(z))).$$

And we notice that g_1 is a translation, g_2 is a rotation, and g_3 is a dilation.

53. If a real number z is positive then we can express it in the polar form $z = r = r(\cos(0) + i\,\sin(0))$ for $r>0$. And it follows that $Arg z = 0$. If a real number z is negative then we can express it in the polar form $z = -r = r(\cos(\pi) + i\,\sin(\pi))$ for $r>0$. And it follows that $Arg z = \pi$. So, it follows that the image of the set S is two numbers 0 and π.

57. We need to solve the equation $\frac{1}{z} = z$ which is equivalent to $z^2 = 1$. The solutions of the last equation are the square roots of 1 which are $+1$ and -1. Hence the fixed points are ± 1.

61. (a) We know that w is in $f[L]$. This means that there is $z = m + i\,n$ with integers m and n such that $z^2 = w$. By the way we can compute

$$w = z^2 = (m + i\,n)^2 = m^2 + 2i\,mn + (i\,n)^2 = m^2 - n^2 + 2i\,mn$$

Now take $m_1 = -n$ and $n_1 = m$. Then for $z_1 = m_1 + i\,n_1$ we compute

$$\begin{aligned} z_1^2 &= (m_1 + i\,n_1)^2 = m_1^2 - n_1^2 + 2i\,m_1 n_1 = (-n)^2 - m^2 + 2i\,(-m)n \\ &= -(m^2 - n^2 + 2i\,mn) = -w. \end{aligned}$$

Hence $-w$ is also from $f[L]$.

If we take $z_2 = \overline{z} = m - i\,n$ then we find

$$z_2^2 = (\overline{z})^2 = \overline{z^2} = \overline{w}.$$

Hence $\overline{w}$ is from $f[L]$.

Finally, we can compute $-\operatorname{Re} w + i\ \operatorname{Im} w = -(\operatorname{Re} w - i\ \operatorname{Im} w) = -\overline{w}$. Since we already proved $u = \overline{w}$ is from $f[L]$. And, for u from $f[L]$ we have $(-u)$ is from $f[L]$ as well. We can conclude that $-\operatorname{Re} w + i\ \operatorname{Im} w = -u$ is from $f[L]$.
(b) This part follows from the formula given in the proof of the part (a). That is for $z = m + i\,n$

$$w = z^2 = (m + i\,n)^2 = m^2 + 2i\,mn + (i\,n)^2 = (m^2 - n^2) + i(2\,mn).$$

We notice that $(m^2 - n^2)$ is integer and $(2mn)$ is integer. So, $w = z^2$ is also from L.
(c) By the part (b) it follows that w is in L. Hence the function f maps the number w to $f(w)$ which is already in $f[L]$. And this is what we needed to prove.

Solutions to Exercises 1.5

1. Note that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{i \sin\left(n\frac{\pi}{2}\right)}{n} \right| \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0.

5. Note that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{\cos n - in}{n^2} \right| \le \lim_{n\to\infty} \frac{|\cos n| + |in|}{n^2} \le \lim_{n\to\infty} \frac{1+n}{n^2} = 0$$

Therefore the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0.

9. (a) Let $L = \lim_{n\to\infty} a_n$. Then given $\epsilon > 0$ we have $|a_n - L| < \epsilon$ for $n > N_\epsilon$. Then $|a_{n+1} - L| < \epsilon$ for $n > N_\epsilon - 1$. Therefore $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$.
(b) We are given $a_1 = i$ and

$$\begin{aligned}
a_{n+1} &= \frac{3}{2+a_n} \\
\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{3}{2+a_n} \\
\lim_{n\to\infty} a_{n+1} &= \frac{3}{2+\lim_{n\to\infty} a_n} \\
L &= \frac{3}{2+L} \qquad \text{(by part (a))} \\
L^2 + 2L - 3 &= 0 \\
(L+3)(L-1) &= 0 \\
L &= 1, -3
\end{aligned}$$

We shall now show that $L = -3$ is absurd.

Claim : $\text{Re}(a_n) \ge 0$

Proof. We shall prove the statement by induction on n. Since $\text{Re}(a_1) = 0$, the statement is true for $n = 1$. Hence assume that the statement is true for some $n \ge 1$. Then we have

$$\begin{aligned}
\text{Re}\,(a_{n+1}) &= \text{Re}\left(\frac{3}{2+a_n}\right) \\
&= \frac{3}{\text{Re}\,(2+a_n)} \\
&= \frac{3}{2+\text{Re}\,(a_n)} \\
&\ge 0
\end{aligned}$$

which completes the proof.
Thus the claim rules out the possibility of $L = -3$. Hence $\lim_{n\to\infty} a_n = 1$.

13. We have

$$\begin{aligned}
\sum_{n=3}^{\infty}\frac{3-i}{(1+i)^n} &= \frac{3-i}{(1+i)^3}\left(\sum_{n=0}^{\infty}\frac{1}{(1+i)^n}\right) && \left(\text{convergent since } \left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{2}} < 1\right)\\
&= \frac{(3-i)}{(1+i)^3}\left(\frac{1}{1-\left(\frac{1}{1+i}\right)}\right) && (\text{geometric series formula})\\
&= \frac{(3-i)}{(1+i)^3}.\frac{(1+i)}{i}\\
&= \frac{(3-i)}{i(1+i)^2}\\
&= -\frac{3}{2}+\frac{i}{2}
\end{aligned}$$

17. Let $S_N = \sum_{n=2}^{N}\frac{1}{(n+i)(n-1+i)}$ denote the N-th partial sum. Then we have

$$\begin{aligned}
\sum_{n=2}^{N}\frac{1}{(n+i)(n-1+i)} &= \sum_{n=2}^{N}\frac{(n+i)-(n-1+i)}{(n+i)(n-1+i)}\\
&= \sum_{n=2}^{N}\left(\frac{1}{(n-1)+i}-\frac{1}{n+i}\right)\\
&= \frac{1}{1+i}-\frac{1}{N+i}
\end{aligned}$$

Note that

$$\lim_{N\to\infty} S_N = \frac{1}{1+i} - \lim_{N\to\infty}\frac{1}{N+i} = \frac{1}{1+i}$$

The series $\sum_{n=2}^{\infty}\frac{1}{(n+i)(n-1+i)}$ is convergent since the partial sums S_N converge and $\sum_{n=2}^{\infty}\frac{1}{(n+i)(n-1+i)} = \lim_{N\to\infty} S_N = \frac{1}{1+i}$.

21. The series $\sum_{n=0}^{\infty}\left(\frac{1+3i}{4}\right)^n$ is convergent because it is a geometric series and the modulus of the common ratio is $\left|\frac{1+3i}{4}\right| = \frac{\sqrt{10}}{4} < 1$.

25. We apply root test to the series $\sum_{n=0}^{\infty}\left(\frac{1+2in}{n}\right)^n$. Note that

$$\rho = \lim_{n\to\infty}\left|\left(\frac{1+2in}{n}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{1+2in}{n}\right| = \lim_{n\to\infty}\left|\frac{1}{n}+2i\right| = \lim_{n\to\infty}\sqrt{4+\frac{1}{n^2}} = 2 > 1$$

Since $\rho > 1$ we conclude that the series $\sum_{n=0}^{\infty}\left(\frac{1+2in}{n}\right)^n$ is divergent.

29. Note that

$$\frac{e^n - ie^{-n}}{e^{n^2}} = e^{n-n^2} - ie^{-n-n^2} = e^{n(1-n)} - ie^{n(-1-n)}$$

Using the root test on $\sum_{n=1}^{\infty} e^{n(1-n)}$ gives

$$\lim_{n\to\infty} \sqrt[n]{e^{n(1-n)}} = \lim_{n\to\infty} e^{1-n} = 0 < 1.$$

Similarly, for $\sum_{n=1}^{\infty} e^{n(-1-n)}$, we have

$$\lim_{n\to\infty} \sqrt[n]{e^{n(-1-n)}} = \lim_{n\to\infty} e^{-1-n} = 0 < 1,$$

so both of these series converge. Thus,

$$\sum_{n=1}^{\infty} \frac{e^n - ie^{-n}}{e^{n^2}} = \sum_{n=1}^{\infty} e^{n(1-n)} - i\sum_{n=1}^{\infty} e^{n(-1-n)}$$

converges.

33. Note that $\sum_{n=0}^{\infty} \frac{z^n}{2^n}$ is a geometric series and it converges iff

$$\begin{aligned} \left|\frac{z}{2}\right| &< 1 \\ |z| &< 2 \end{aligned}$$

Now if z satisfies the inequality $|z| < 2$ we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^n}{2^n} &= \frac{1}{1 - \left(\frac{z}{2}\right)} \qquad \text{(geometric series formula)} \\ &= \frac{2}{2-z}. \end{aligned}$$

37. Note that $\sum_{n=1}^{\infty} \frac{1}{(2-10z)^n}$ is a geometric series and it converges iff

$$\begin{aligned} \left|\frac{1}{2-10z}\right| &< 1 \\ |2-10z| &> 1 \\ \left|z - \frac{1}{5}\right| &> \frac{1}{10} \qquad \text{(dividing both sides by 10)} \end{aligned}$$

Now if z satisfies the inequality $\left|z-\dfrac{1}{5}\right|>\dfrac{1}{10}$ we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(2-10z)^n}&=\left(\frac{1}{2-10z}\right)\left(\sum_{n=0}^{\infty}\frac{1}{(2-10z)^n}\right)\\&=\left(\frac{1}{2-10z}\right)\left(\frac{1}{1-\dfrac{1}{2-10z}}\right)\qquad\text{(geometric series formula)}\\&=\frac{1}{1-10z}.\end{aligned}$$

41. We are given that the n-th partial sum $s_n=\dfrac{i}{n}$. Thus

$$\left|\lim_{n\to\infty}s_n\right|=\left|\lim_{n\to\infty}\frac{i}{n}\right|=\lim_{n\to\infty}\frac{1}{n}=0.$$

Therefore the series converges to 0.

45. We have

$$\begin{aligned}a_{n+1}&=\frac{(7+3i)n}{1+2in^2}a_n\\\frac{a_{n+1}}{a_n}&=\frac{(7+3i)n}{1+2in^2}\\\left|\frac{a_{n+1}}{a_n}\right|&=\left|\frac{(7+3i)n}{1+2in^2}\right|=\frac{n\sqrt{58}}{\sqrt{1+4n^4}}=\frac{\sqrt{58}}{n\sqrt{4+\dfrac{1}{n^4}}}\\\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|&=\lim_{n\to\infty}\frac{\sqrt{58}}{n\sqrt{4+\dfrac{1}{n^4}}}\\&=0<1\end{aligned}$$

Hence by the ratio test we conclude that the series $\sum_{n=0}^{\infty}a_n$ converges.

Solutions to Exercises 1.6

1. (a) We have
$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

(b) We have
$$e^{2i\pi} = \cos(2\pi) + i\sin(2\pi) = 1.$$

(c) We have
$$3e^{-1+200i\pi} = 3e^{-1}\cdot e^{200i\pi} = 3e^{-1}\left(\cos(200\pi) + i\sin(200\pi)\right) = 3e^{-1}\cdot 1 = \frac{3}{e}.$$

(d) We have
$$\frac{e^{\ln(3)+201i\frac{\pi}{2}}}{3} = \frac{e^{\ln(3)}\cdot e^{201i\frac{\pi}{2}}}{3} = \frac{3\left(\cos\left(201\frac{\pi}{2}\right) + i\sin\left(201\frac{\pi}{2}\right)\right)}{3} = i.$$

5. (a) We have
$$\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}.$$

(b) We have
$$\sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right) = e^{i\left(\frac{\pi}{2}-\theta\right)}.$$

(c) We have
$$\frac{1}{\cos\theta + i\sin\theta} = \frac{1}{e^{i\theta}} = e^{-i\theta}.$$

(d) We have
$$\frac{\cos\theta + i\sin\theta}{\cos(3\theta) + i\sin(3\theta)} = \frac{e^{i\theta}}{e^{3i\theta}} = e^{i\theta-3i\theta} = e^{-2i\theta}.$$

9. (a) We have
$$e^{z_1} = e^{1+i} = e^1\cdot e^i = e\cdot(\cos 1 + i\sin 1) = e\cos(1) + ie\sin(1).$$

(b) We have
$$3ie^{z_2} = 3ie^{1-i} = 3ie^1\cdot e^{-i} = 3ie(\cos(-1) + i\sin(-1)) = 3e\sin(1) + i3e\cos(1).$$

(c) We have
$$e^{z_1}.e^{z_2} = e^{z_1+z_2} = e^{(1+i)+(1-i)} = e^2.$$

(d) We have
$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2} = e^{(1+i)-(1-i)} = e^{2i} = \cos(2) + i\sin(2).$$

13. (a) We have
$$-3-3i = 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 3\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) = 3\sqrt{2}e^{i\frac{5\pi}{4}}.$$

(b) We have

$$-\frac{\sqrt{3}}{2}+i\frac{1}{2}=\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)=e^{i\frac{5\pi}{6}}.$$

(c) We have

$$-1-\sqrt{3}i=2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=2\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)=2e^{i\frac{4\pi}{3}}.$$

(d) We have

$$-3e^{2i}=3(-\cos(2)-i\sin(2))=3(\cos(\pi+2)+i\sin(\pi+2))=3e^{i(\pi+2)}.$$

17. Consider a vertical line segment of S,

$$L_{x_0}=\left\{x_0+iy:0\le y\le\frac{\pi}{2}\right\}.$$

An arbitrary point $z=x_0+iy$ of L_{x_0} is mapped to

$$f(z)=e^z=e^{x_0+iy}=e^{x_0}\cdot e^{iy}.$$

Now as y varies from 0 to $\frac{\pi}{2}$, the image $f(z)$ traces a quarter circle having radius e^{x_0}. Thus we get

$$f(L_{x_0})=\left\{e^{x_0}e^{i\theta}:0\le\theta\le\frac{\pi}{2}\right\}.$$

Therefore

$$\begin{aligned}
f(S)&=f\left(\bigcup_{-3\le x_0\le 3}L_{x_0}\right)\\
&=\bigcup_{-3\le x_0\le 3}f(L_{x_0})\\
&=\bigcup_{-3\le x_0\le 3}\left\{e^{x_0}e^{i\theta}:0\le\theta\le\frac{\pi}{2}\right\}\\
&=\left\{re^{i\theta}:e^{-3}\le r\le e^3,\ 0\le\theta\le\frac{\pi}{2}\right\}
\end{aligned}$$

21. Consider a vertical line segment of S,

$$L_{x_0}=\{x_0+iy:0\le y\le\pi\}$$

An arbitrary point $z=x_0+iy$ of L_{x_0} is mapped to

$$f(z)=e^z=e^{x_0+iy}=e^{x_0}\cdot e^{iy}$$

Now as y varies from 0 to π, the image $f(z)$ traces a semicircle having radius e^{x_0}. Thus we get

$$f[L_{x_0}]=\left\{e^{x_0}e^{i\theta}:0\le\theta\le\pi\right\}.$$

Therefore

$$\begin{aligned} f[S] &= f\left[\bigcup_{-\infty<x_0<\infty} L_{x_0}\right] \\ &= \bigcup_{-\infty<x_0<\infty} f[L_{x_0}] \\ &= \bigcup_{-\infty<x_0<\infty} \left\{e^{x_0}e^{i\theta} : 0 \le \theta \le \pi\right\} \\ &= \{z \in \mathbb{C} : \operatorname{Im}(z) \ge 0\} \setminus \{0\}. \end{aligned}$$

25. (a) We have

$$\begin{aligned} e^z &= 2 - 2i \\ e^z &= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \\ e^z &= 2\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right) \\ e^z &= e^{\ln\left(2\sqrt{2}\right)} \cdot e^{i\frac{7\pi}{4}} \\ e^{z-\ln\left(2\sqrt{2}\right)-i\frac{7\pi}{4}} &= 1 \\ z - \ln\left(2\sqrt{2}\right) - i\frac{7\pi}{4} &= i2k\pi \qquad (k \in \mathbb{Z}) \\ z &= \ln\left(2\sqrt{2}\right) + i\left(2k + \frac{7}{4}\right)\pi \qquad (k \in \mathbb{Z}). \end{aligned}$$

(b) We have

$$\begin{aligned} e^{2z} &= i \\ e^{2z} &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ e^{2z} &= e^{i\frac{\pi}{2}} \\ e^{2z-i\frac{\pi}{2}} &= 1 \\ 2z - i\frac{\pi}{2} &= i2k\pi \qquad (k \in \mathbb{Z}) \\ z &= i\left(k + \frac{1}{4}\right)\pi \qquad (k \in \mathbb{Z}) \end{aligned}$$

29. Let $z = x + iy$ and we have

$$\begin{aligned} |e^z| \le 1 &\iff e^x \le 1 \qquad \left(|e^{x+iy}| = e^x, \text{ see Exercise 33.}\right) \\ &\iff x < 0 \\ &\iff \operatorname{Re}(z) < 0 \end{aligned}$$

Solutions to Exercises 1.7

1. (a)

$$\begin{aligned}\cos i &= \frac{e^{i(i)}+e^{-i(i)}}{2}\\ &= \frac{e^{-1}+e^{1}}{2}\\ &= \cosh(1)\\ &= \cos(0)\cosh(1) - i\sin(0)\sinh(1)\end{aligned}$$

$$\begin{aligned}\sin i &= \frac{e^{i(i)}-e^{-i(i)}}{2i}\\ &= \frac{e^{-1}-e^{1}}{2i}\\ &= i\left(\frac{e^{1}-e^{-1}}{2}\right)\\ &= i\sinh(1)\\ &= \sin(0)\cosh(1) + i\cos(0)\sinh(1)\end{aligned}$$

(b)

$$\begin{aligned}\cos\frac{\pi}{2} &= 0\\ &= \cos\left(\frac{\pi}{2}\right)\cosh(0) - i\sin\left(\frac{\pi}{2}\right)\sinh(0)\end{aligned}$$

$$\begin{aligned}\sin\frac{\pi}{2} &= 1\\ &= \sin\left(\frac{\pi}{2}\right)\cosh(0) + i\cos\left(\frac{\pi}{2}\right)\sinh(0)\end{aligned}$$

(c)

$$\begin{aligned}\cos(\pi+i) &= \frac{e^{i(\pi+i)}+e^{-i(\pi+i)}}{2}\\ &= \frac{e^{i\pi}\cdot e^{-1}+e^{-i\pi}\cdot e^{1}}{2}\\ &= \frac{(\cos\pi + i\sin\pi)\cdot e^{-1} + (\cos\pi - i\sin\pi)\cdot e^{1}}{2}\\ &= \left(\frac{e^{1}+e^{-1}}{2}\right)\cos\pi - i\left(\frac{e^{1}-e^{-1}}{2}\right)\sin\pi\\ &= \cos(\pi)\cosh(1) - i\sin(\pi)\sinh(1)\end{aligned}$$

$$\begin{aligned}
\sin(\pi+i) &= \frac{e^{i(\pi+i)}-e^{-i(\pi+i)}}{2i} \\
&= \frac{e^{i\pi}\cdot e^{-1}-e^{-i\pi}\cdot e^{1}}{2i} \\
&= \frac{(\cos\pi+i\sin\pi)\cdot e^{-1}-(\cos\pi-i\sin\pi)\cdot e^{1}}{2i} \\
&= \left(\frac{e^{1}+e^{-1}}{2}\right)\sin\pi+i\left(\frac{e^{1}-e^{-1}}{2}\right)\cos\pi \\
&= \sin(\pi)\cosh(1)+i\cos(\pi)\sinh(1)
\end{aligned}$$

(d)

$$\begin{aligned}
\cos\Big(\frac{\pi}{2}+2\pi i\Big) &= \frac{e^{i(\frac{\pi}{2}+2\pi i)}+e^{-i(\frac{\pi}{2}+2\pi i)}}{2} \\
&= \frac{e^{\frac{\pi}{2}i}\cdot e^{-2\pi}+e^{-\frac{\pi}{2}i}\cdot e^{2\pi}}{2} \\
&= \frac{(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})\cdot e^{-2\pi}+(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2})\cdot e^{2\pi}}{2} \\
&= \left(\frac{e^{2\pi}+e^{-2\pi}}{2}\right)\cos\frac{\pi}{2}-i\left(\frac{e^{2\pi}-e^{-2\pi}}{2}\right)\sin\frac{\pi}{2} \\
&= \cos\Big(\frac{\pi}{2}\Big)\cosh(2\pi)-i\sin\Big(\frac{\pi}{2}\Big)\sinh(2\pi)
\end{aligned}$$

$$\begin{aligned}
\sin\Big(\frac{\pi}{2}+2\pi i\Big) &= \frac{e^{i(\frac{\pi}{2}+2\pi i)}-e^{-i(\frac{\pi}{2}+2\pi i)}}{2i} \\
&= \frac{e^{\frac{\pi}{2}i}\cdot e^{-2\pi}-e^{-\frac{\pi}{2}i}\cdot e^{2\pi}}{2i} \\
&= \frac{(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})\cdot e^{-2\pi}-(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2})\cdot e^{2\pi}}{2i} \\
&= \left(\frac{e^{2\pi}+e^{-2\pi}}{2}\right)\sin\frac{\pi}{2}+i\left(\frac{e^{2\pi}-e^{-2\pi}}{2}\right)\cos\frac{\pi}{2} \\
&= \sin\Big(\frac{\pi}{2}\Big)\cosh(2\pi)+i\cos\Big(\frac{\pi}{2}\Big)\sinh(2\pi)
\end{aligned}$$

5.

$$\begin{aligned}
\cos(1+i) &= \cos(1)\cosh(1)-i\sin(1)\sinh(1) \\
\sin(1+i) &= \sin(1)\cosh(1)+i\cos(1)\sinh(1) \\
\tan(1+i) &= \frac{\sin(1+i)}{\cos(1+i)} \\
&= \frac{\sin(1)\cosh(1)+i\cos(1)\sinh(1)}{\cos(1)\cosh(1)-i\sin(1)\sinh(1)} \\
|\cos(1+i)| &= \sqrt{\cos^2(1)+\sinh^2(1)} \\
|\sin(1+i)| &= \sqrt{\sin^2(1)+\sinh^2(1)}
\end{aligned}$$

9.

$$\begin{aligned}\sin(2z) &= \sin(2(x+iy))\\ &= \sin(2x+i2y)\\ &= \sin(2x)\cosh(2y)+i\cos(2x)\sinh(2y)\end{aligned}$$

13.

$$\begin{aligned}\tan z &= \frac{\sin z}{\cos z}\\ &= \frac{\sin(x+iy)}{\cos(x+iy)}\\ &= \frac{\sin(x)\cosh(y)+i\cos(x)\sinh(y)}{\cos(x)\cosh(y)-i\sin(x)\sinh(y)}\\ &= \left(\frac{\sin(x)\cosh(y)+i\cos(x)\sinh(y)}{\cos(x)\cosh(y)-i\sin(x)\sinh(y)}\right)\cdot\left(\frac{\cos(x)\cosh(y)+i\sin(x)\sinh(y)}{\cos(x)\cosh(y)+i\sin(x)\sinh(y)}\right)\\ &= \frac{\sin(x)\cos(x)\cosh^2 y-\sin(x)\cos(x)\sinh^2 y+i(\sinh(y)\cosh(y)\sin^2(x)+\sinh(y)\cosh(y)\cos^2(x))}{\cos^2 x+\sinh^2 y}\\ &= \frac{\sin(x)\cos(x)+i\sinh(y)\cosh(y)}{\cos^2 x+\sinh^2 y}\\ &= \left(\frac{\sin(x)\cos(x)}{\cos^2 x+\sinh^2 y}\right)+i\left(\frac{\sinh(y)\cosh(y)}{\cos^2 x+\sinh^2 y}\right)\end{aligned}$$

17. Consider the horizontal strip

$$L_{y_0}=\left\{x+iy_0 : -\frac{\pi}{2}\le x\le\frac{\pi}{2}\right\}$$

The mapping $f(z)=\sin z$ maps an arbitrary point of L_{y_0} to

$$f(x+iy_0)=\sin(x+iy_0)=\sin x\cosh y_0+i\cos x\sinh y_0$$

Set $u=\sin x\cosh y_0$ and $v=\cos x\sinh y_0$. Observe that

$$\left(\frac{u}{\cosh y_0}\right)^2+\left(\frac{v}{\sinh y_0}\right)^2=1$$

Now as x varies in the interval $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, the point (u,v) traces an upper semi-ellipse with u-intercepts $(\pm\cosh y_0,0)$ and v-intercept $(0,\sinh y_0)$. Thus

$$f[L_{y_0}]=\left\{u+iv : \left(\frac{u}{\cosh y_0}\right)^2+\left(\frac{v}{\sinh y_0}\right)^2=1,\ v\ge 0\right\}$$

Thus

$$\begin{aligned}
f[S] &= f\Bigg[\bigcup_{\alpha\le y_0\le\beta} L_{y_0}\Bigg]\\
&= \bigcup_{\alpha\le y_0\le\beta} f[L_{y_0}]\\
&= \bigcup_{\alpha\le y_0\le\beta}\left\{u+iv : \left(\frac{u}{\cosh y_0}\right)^2 + \left(\frac{v}{\sinh y_0}\right)^2 = 1,\ v\ge 0\right\}\\
&= \left\{u+iv : \left(\frac{u}{\cosh\alpha}\right)^2 + \left(\frac{v}{\sinh\alpha}\right)^2 \ge 1,\ \left(\frac{u}{\cosh\beta}\right)^2 + \left(\frac{v}{\sinh\beta}\right)^2 \le 1\right\}.
\end{aligned}$$

21.

$$\begin{aligned}
\sin z &= \sin(x+iy)\\
&= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}\\
&= \frac{e^{-y+ix} - e^{y-ix}}{2i}\\
&= \frac{e^{-y}\cdot e^{ix} - e^{y}\cdot e^{-ix}}{2i}\\
&= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - \sin x)}{2i}\\
&= \left(\frac{e^{y}+e^{-y}}{2}\right)\sin x + i\left(\frac{e^{y}-e^{-y}}{2}\right)\cos x\\
&= \sin x\cosh y + i\cos x\sinh y
\end{aligned}$$

$$\begin{aligned}
|\sin z| &= |\sin(x+iy)|\\
&= |\sin x\cosh y + i\cos x\sinh y|\\
&= \sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}\\
&= \sqrt{\sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y}\\
&= \sqrt{\sin^2 x + \sinh^2 y}.
\end{aligned}$$

25.

$$\begin{aligned}
\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}\\
\cos^3\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3\\
\cos^3\theta &= \frac{e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta}}{8}\\
\cos^3\theta &= \frac{\cos(3\theta)+i\sin(3\theta)+\cos(\theta)+i\sin(\theta)+\cos(\theta)-i\sin(\theta)+\cos(3\theta)-i\sin(3\theta)}{8}\\
\cos^3\theta &= \frac{\cos 3\theta+3\cos\theta}{4}
\end{aligned}$$

29.

$$\begin{aligned}
&\sin z_1\cos z_2+\cos z_1\sin z_2\\
&=\left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right)+\left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)\\
&=\frac{e^{i(z_1+z_2)}+e^{i(z_1-z_2)}-e^{i(z_2-z_1)}-e^{-i(z_1+z_2)}+e^{i(z_1+z_2)}-e^{i(z_1-z_2)}+e^{i(z_2-z_1)}-e^{-i(z_1+z_2)}}{4i}\\
&=\frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}\\
&=\sin(z_1+z_2)
\end{aligned}$$

33.

$$\begin{aligned}
2\sin z_1\sin z_2 &= 2\left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)\\
&=-\left(\frac{e^{i(z_1+z_2)}-e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}+e^{-i(z_1+z_2)}}{2}\right)\\
&=\frac{e^{i(z_1-z_2)}+e^{-i(z_1-z_2)}}{2}-\frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2}\\
&=\cos(z_1-z_2)-\cos(z_1+z_2).
\end{aligned}$$

37.

$$\begin{aligned}
\cosh(z+\pi i) &= \cos(i(z+\pi i))\\
&=\cos(iz-\pi)\\
&=-\cos(iz)\\
&=-\cosh z
\end{aligned}$$

$$\begin{aligned}
\sinh(z+\pi i) &= -i\sin(i(z+\pi i)) \\
&= -i\sin(iz-\pi) \\
&= i\sin(iz) \\
&= -\sinh z.
\end{aligned}$$

41.

$$\begin{aligned}
1+2\sinh^2 z &= 1+2(-i\sin(iz))^2 \\
&= 1-2\sin^2(iz) \\
&= 2\cos^2(iz)-1 \\
&= 2\cosh^2 z-1 \\
&= 2\cos^2(iz)-1 \\
&= \cos(2iz) \\
&= \cosh(2z) \\
&= \cos(2iz) \\
&= \cos^2(iz)-\sin^2(iz) \\
&= (\cos(iz))^2+(-i\sin(iz))^2 \\
&= \cosh^2 z+\sinh^2 z.
\end{aligned}$$

45.

$$\begin{aligned}
&\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \\
&= \cos(iz_1)\cos(iz_2) + (-i\sin(iz_1))(-i\sin(iz_2)) \\
&= \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2) \\
&= \cos(iz_1 - iz_2) \\
&= \cos(i(z_1 - z_2)) \\
&= \cosh(z_1 - z_2).
\end{aligned}$$

49.

$$\begin{aligned}
&2\sinh z_1 \cosh z_2 \\
&= 2(-i\sin(iz_1))(\cos(iz_2)) \\
&= -i(2\sin(iz_1)\cos(iz_2)) \\
&= -i(\sin(iz_1+iz_2)+\sin(iz_1-iz_2)) \\
&= -i\sin(i(z_1+z_2)) - i\sin(i(z_1-z_2)) \\
&= \sinh(z_1+z_2)+\sinh(z_1-z_2).
\end{aligned}$$

53. (a) Let

$$S = 1 + z + \cdots + z^n.$$

Then

$$zS = z + z^2 + \cdots + z^{n+1}.$$

By subtracting the second equation from the first, we get

$$\begin{aligned} S - zS &= (1 + z + \cdots + z^n) - (z + z^2 + \cdots + z^{n+1}) \\ (1-z)S &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1). \end{aligned}$$

(b)

$$\begin{aligned} 1 + e^{i\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \qquad \text{(By Problem 53(a).)} \\ &= \left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right)\left(\frac{e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}}\right) \\ &= \frac{(1 - e^{i(n+1)\theta}) \cdot e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}} \\ &= \frac{\left(\dfrac{(e^{i(n+1)\theta} - 1) \cdot e^{-\frac{i\theta}{2}}}{2i}\right)}{\left(\dfrac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{2i}\right)} \\ &= \frac{i(1 - e^{i(n+1)\theta}) \cdot e^{-\frac{i\theta}{2}}}{2\sin\frac{\theta}{2}} \end{aligned}$$

(c) From part (b), we get

$$\begin{aligned} 1 + e^{i\theta} + \cdots + e^{in\theta} &= \frac{i(1 - e^{i(n+1)\theta}) \cdot e^{-\frac{i\theta}{2}}}{2\sin\frac{\theta}{2}} \\ &= \frac{i(e^{-\frac{i\theta}{2}} - e^{i(n+\frac{1}{2})\theta})}{2\sin\frac{\theta}{2}} \\ &= \frac{i\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} - \cos\big(n+\frac{1}{2}\big)\theta - i\sin\big(n+\frac{1}{2}\big)\theta\right)}{2\sin\frac{\theta}{2}} \\ &= \left(\frac{1}{2} + \frac{\sin\big(n+\frac{1}{2}\big)\theta}{2\sin\frac{\theta}{2}}\right) + i\left(\frac{\cos\frac{\theta}{2} - \cos\big(n+\frac{1}{2}\big)\theta}{2\sin\frac{\theta}{2}}\right) \end{aligned}$$

By taking the real and imaginary parts of the above identity, we get

$$
\begin{aligned}
\operatorname{Re}\left(1 + e^{i\theta} + \cdots + e^{in\theta}\right) &= \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \\
1 + \cos\theta + \cdots + \cos n\theta &= \frac{1}{2} + \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \\
\frac{1}{2} + \cos\theta + \cdots + \cos n\theta &= \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Im}\left(1 + e^{i\theta} + \cdots + e^{in\theta}\right) &= \frac{\cos\frac{\theta}{2} - \cos\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} \\
\sin\theta + \cdots + \sin n\theta &= \frac{\cos\frac{\theta}{2} - \cos\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}
\end{aligned}
$$

Solutions to Exercises 1.8

1. (a)

$$\begin{aligned}\log(2i) &= \ln(|2i|) + i\arg(2i)\\ &= \ln 2 + i\left(\frac{\pi}{2} + 2k\pi\right) \qquad (k \in \mathbb{Z}).\end{aligned}$$

(b)

$$\begin{aligned}\log(-3-3i) &= \ln(|-3-3i|) + i\arg(-3-3i)\\ &= \ln\left(3\sqrt{2}\right) + i\left(\frac{5\pi}{4} + 2k\pi\right) \qquad (k \in \mathbb{Z}).\end{aligned}$$

(c)

$$\begin{aligned}\log\left(5e^{i\frac{\pi}{7}}\right) &= \ln\left(|5e^{i\frac{\pi}{7}}|\right) + i\arg(5e^{i\frac{\pi}{7}})\\ &= \ln 5 + i\left(\frac{\pi}{7} + 2k\pi\right) \qquad (k \in \mathbb{Z}).\end{aligned}$$

(d)

$$\begin{aligned}\log(-3) &= \ln(|-3|) + i\arg(-3)\\ &= \ln 3 + i(\pi + 2k\pi) \qquad (k \in \mathbb{Z}).\end{aligned}$$

5. If we know $\log z$, to find $\operatorname{Log} z$, it suffices to choose the value of $\log z$ with the imaginary part lying in the interval $(-\pi, \pi]$.

$$\begin{aligned}\operatorname{Log}(2i) &= \ln 2 + i\left(\frac{\pi}{2}\right)\\ \operatorname{Log}(-3-3i) &= \ln\left(3\sqrt{2}\right) - i\left(\frac{3\pi}{4}\right)\\ \operatorname{Log}(5e^{i\frac{\pi}{7}}) &= \ln 5 + i\left(\frac{\pi}{7}\right)\\ \operatorname{Log}(-3) &= \ln 3 + i\pi.\end{aligned}$$

9. Note that

$$\log 1 = \ln|1| + i\arg(1) = i2k\pi \qquad (k \in \mathbb{Z})$$

If we know $\log z$, to find $\log_6 z$, it suffices to choose the value of $\log z$ with the imaginary part lying in the interval $(6, 6+2\pi]$. Thus

$$\log_6 1 = i2\pi.$$

13. We have

$$\begin{aligned}
e^z &= 3\\
e^z &= e^{\operatorname{Log} 3}\\
e^{z-\operatorname{Log} 3} &= 1\\
z - \operatorname{Log} 3 &= i2k\pi \qquad (k \in \mathbb{Z})\\
z &= \operatorname{Log} 3 + i2k\pi \qquad (k \in \mathbb{Z})\\
&= \ln 3 + i2k\pi \qquad (k \in \mathbb{Z})
\end{aligned}$$

17. We have

$$\begin{aligned}
e^{2z} + 5 &= 0\\
e^{2z} &= -5\\
e^{2z} &= e^{\operatorname{Log}(-5)}\\
e^{2z-\operatorname{Log}(-5)} &= 1\\
2z - \operatorname{Log}(-5) &= i2k\pi \qquad (k \in \mathbb{Z})\\
z &= \frac{1}{2}\operatorname{Log}(-5) + ik\pi \qquad (k \in \mathbb{Z})\\
&= \tfrac{1}{2}\ln 5 + i\left(\frac{\pi}{2} + k\pi\right) \qquad (k \in \mathbb{Z}).
\end{aligned}$$

21.

$$\begin{aligned}
\operatorname{Log}(-1) &= \ln|-1| + i\operatorname{Arg}(-1) = i\pi\\
\operatorname{Log}(i) &= \ln|i| + i\operatorname{Arg}(i) = i\left(\frac{\pi}{2}\right)\\
\operatorname{Log}(-i) &= \ln|-i| + i\operatorname{Arg}(-i) = -i\left(\frac{\pi}{2}\right).
\end{aligned}$$

Note that

$$\operatorname{Log}(-1)(i) = \operatorname{Log}(-i) = -i\left(\frac{\pi}{2}\right) \neq i(\pi) + i\left(\frac{\pi}{2}\right) = \operatorname{Log}(-1) + \operatorname{Log}(i).$$

25.

$$\begin{aligned}
\operatorname{Log} z + \operatorname{Log}(2z) &= \frac{3\pi}{2}\\
\ln|z| + i\operatorname{Arg}(z) + \ln|2z| + i\operatorname{Arg}(2z) &= \frac{3\pi}{2}\\
\ln|2z^2| + i2\operatorname{Arg}(z) &= \frac{3\pi}{2} \qquad (\operatorname{Arg}(z) = \operatorname{Arg}(2z))
\end{aligned}$$

By comparing the real and imaginary parts, we get two relations

$$\begin{aligned}
\ln|2z^2| &= \frac{3\pi}{2}\\
2\operatorname{Arg}(z) &= 0
\end{aligned}$$

From the first relation we get

$$\begin{aligned}\ln|2z^2| &= \frac{3\pi}{2}\\ 2|z|^2 &= e^{\frac{3\pi}{2}}\\ |z| &= \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}\end{aligned}$$

From the second relation we get

$$\begin{aligned}2\,\mathrm{Arg}\,(z) &= 0\\ \mathrm{Arg}\,z &= 0\end{aligned}$$

Thus $z = |z|.\,(\cos(\,\mathrm{Arg}\,z) + i\sin(\,\mathrm{Arg}\,z)) = \dfrac{e^{\frac{3\pi}{4}}}{\sqrt{2}}.$

29.

$$\begin{aligned}5^i &= e^{i\,\mathrm{Log}\,5}\\ &= e^{i(\ln|5|+i\,\mathrm{Arg}\,(5))}\\ &= e^{i\ln 5}\end{aligned}$$

33.

$$\begin{aligned}(3i)^4 &= 3^4\cdot i^4\\ &= 81\end{aligned}$$

Thus $(3i)^4 = 81$ has a unique value.

37. Set $z = -1$. Then

$$\mathrm{Log}\,\overline{z} = \mathrm{Log}\,\overline{-1} = \mathrm{Log}\,(-1) = \ln|-1| + i\,\mathrm{Arg}\,(-1) = i\pi$$

Again

$$\overline{\mathrm{Log}\,z} = \overline{\mathrm{Log}\,(-1)} = \overline{\ln|-1| + i\,\mathrm{Arg}\,(-1)} = \overline{i\pi} = -i\pi$$

Thus we see that

$$\mathrm{Log}\,(\overline{-1}) \neq \overline{\mathrm{Log}\,(-1)}$$

41. By problem 38, the image of the punctured plane $\mathbb{C}\setminus\{0\}$ under the mapping $f(z) = \log_{3\pi} z$ is $S_{3\pi} = \{z = x + iy : 3\pi < y \leq 5\pi\}$.

45. The domain of the map is

$$S = \{z = r\cdot e^{i\theta} : \frac{1}{2} \leq r \leq 1,\ -\pi < \theta \leq \pi\}.$$

The mapping $f(z) = \mathrm{Log}\,z$ maps an arbitrary point $z = r\cdot e^{i\theta}$ of S to

$$f(z) = \mathrm{Log}\,(r\cdot e^{i\theta}) = \ln|r\cdot e^{i\theta}| + i\,\mathrm{Arg}\,(r\cdot e^{i\theta}) = \ln r + i\theta.$$

Set $u = \ln r$ and $v = \theta$. As r varies between $\frac{1}{2} \le r \le 1$, u varies between $-\ln 2 \le u \le 0$ and as θ varies between $-\pi < \theta \le \pi$, v also varies between $-\pi < v \le \pi$. Thus

$$f[S] = \{u + iv : -\ln 2 \le u \le 0,\ -\pi < v \le \pi\}.$$

49. (a) We have

$$\begin{aligned}
\tan w &= \frac{\sin w}{\cos w} \\
\tan w &= \left(\frac{e^{iw} - e^{-iw}}{2i}\right) \cdot \left(\frac{2}{e^{iw} + e^{-iw}}\right) \\
i \tan w &= \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \\
1 + i \tan w &= 1 + \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \\
&= \frac{2e^{iw}}{e^{iw} + e^{-iw}}
\end{aligned}$$

(b) From part (b), we have

$$\begin{aligned}
i \tan w &= \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \\
1 - i \tan w &= 1 - \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \\
&= \frac{2e^{-iw}}{e^{iw} + e^{-iw}}
\end{aligned}$$

(c) From parts (a) and (b) we get

$$\begin{aligned}
\frac{1 + i \tan w}{1 - \tan w} &= \left(\frac{2e^{iw}}{e^{iw} + e^{-iw}}\right) \cdot \left(\frac{e^{iw} + e^{-iw}}{2e^{-iw}}\right) \\
\frac{1 + i \tan w}{1 - \tan w} &= e^{i2w} \\
\frac{1 + iz}{1 - iz} &= e^{i2w} \qquad \text{(Set } z = \tan w\text{)} \\
\log\left(\frac{1 + iz}{1 - iz}\right) &= \log\bigl(e^{i2w}\bigr) \\
\log\left(\frac{1 + iz}{1 - iz}\right) &= i2w \\
w &= \frac{i}{2} \log\left(\frac{1 - iz}{1 + iz}\right)
\end{aligned}$$

53. (a)

$$\begin{aligned}
z^{\frac{p}{q}} &= e^{\frac{p}{q}\log z} \\
&= e^{\frac{p}{q}(\ln|z|+i\arg(z))} \\
&= e^{\frac{p}{q}(\ln|z|+i(\operatorname{Arg}(z)+2k\pi))} \qquad (k \in \mathbb{Z}) \\
&= e^{\frac{p}{q}(\operatorname{Log} z+i2k\pi)} \qquad (k \in \mathbb{Z}) \\
&= e^{\left(\frac{p}{q}\right)\operatorname{Log} z} \cdot e^{\frac{i2kp\pi}{q}} \qquad (k \in \mathbb{Z})
\end{aligned}$$

(b) Set $E_n = e^{\frac{i2np\pi}{q}}$. Now

$$E_{n+q} = e^{\frac{i2(n+q)p\pi}{q}} = e^{\frac{i2np\pi}{q}+i2p\pi} = e^{\frac{i2np\pi}{q}} \cdot e^{i2p\pi} = e^{\frac{i2np\pi}{q}} \cdot 1 = E_n$$

Since $E_n = E_{n+q}$, there can be at most q values for $z^{\frac{p}{q}}$.
(c) Lets suppose that $E_j = E_l$ for some $0 \le j < l \le q-1$. Thus

$$\begin{aligned}
e^{\frac{i2jp\pi}{q}} &= e^{\frac{i2lp\pi}{q}} \\
e^{\frac{i2(l-j)p\pi}{q}} &= 1 \\
\frac{i2(l-j)p\pi}{q} &= i2k\pi \qquad \text{(for some integer } k) \\
\frac{p(l-j)}{q} &= k
\end{aligned}$$

(d) It is impossible for $\frac{p(l-j)}{q}$ to be an integer since $\gcd(p,q) = 1$ and $0 < (l-j) < q$. Therefore E_n are distinct for $0 \le n \le q-1$ and hence $z^{\frac{p}{q}}$ has q distinct values.

Solutions to Exercises 2.1

1. We notice that the set $\{z : |z| \le 1\}$ is the closed disk of radius one centered at the origin. The interior points are $\{z : |z| < 1\}$, or the open unit disk centered at the origin. And, the boundary is the set $\{z : |z| = 1\}$, or the unit circle centered at the origin.

5. We see that the set $\{z : \operatorname{Re} z > 0\}$ is a right half-plane. Since the inequality $\operatorname{Re} z > 0$ is strict, this set is open. Really, if some z_0 is in from the set $\{z : \operatorname{Re} z > 0\}$. Then for any number $0 < r < \operatorname{Re} z$ and any complex number z such that $|z - z_0| < r$ we have $\operatorname{Re} z > \operatorname{Re} z_0 - r > 0$. Therefore z is also from the same set $\{z : \operatorname{Re} z > 0\}$. This justifies that the set $\{z : \operatorname{Re} z > 0\}$ is open. Also, it is clear from its picture that any two points from the set $\{z : \operatorname{Re} z > 0\}$ can be connected even by a straight line segment, the simplest polygonal line. So, the set $\{z : \operatorname{Re} z > 0\}$ is also connected. Since it is simultaneously open and connected this set is a region.

9. We see that the set $A = \{z : z \ne 0, |\operatorname{Arg} z| < \frac{\pi}{4}\} \cup \{0\}$ is an infinite sector with the vertex $z = 0$ included but the rays $\{z : z \ne 0, \operatorname{Arg} z = \pm\frac{\pi}{4}\}$ are not included. Since any open disc with the center at the origin $z = 0$ is not contained in the set A it follows this set is not open. On the other hand since the rays $\{z : z \ne 0, \operatorname{Arg} z = \pm\frac{\pi}{4}\}$ are not in this set A it follows the set A is also not closed. It is clear from the picture that any two points from the set A can be connected by a straight line segment. So, the set A is connected. Since it is not open the set A is not a region.

13. One of simple examples is the following. Let us take the sets $A = \{0\}$ and $B = \{1\}$. Each of these sets is a point. Any single point is a connected set but it is obviously that $A \cup B = \{0\} \cup \{1\}$ consists of only two points and they cannot be connected by a polygonal line inside $A \cup B$.

17. To prove this statement we need to show that if $\mathbb{C} \setminus S$ is closed then any point z of the set S is an interior point of S. Let us assume this is not true and there it is a point z_0 of S which is not an interior point. This means that every neighborhood of z_0 contains at least one point not from S. Thus, every neighborhood of z_0 contains at least one point from the complement of S, the set $\mathbb{C} \setminus S$. And z_0 is not from $\mathbb{C} \setminus S$. By definition of boundary of a set it follows that z_0 is from the boundary of the set $\mathbb{C} \setminus S$. But we know that the set $\mathbb{C} \setminus S$ is closed and thus it contains all its boundary. And, thus z_0 must be in $\mathbb{C} \setminus S$ which is impossible since z_0 is from the set S. This gives a contradiction to our assumption. Hence the assumption was wrong and the set S is open.

21. To show that the set $A \cup B$ is a region we need to show that is open and connected.

Let z_0 is a point of $A \cup B$. Thus, z_0 is either from A or from B. If z_0 is from A then since A is open it follows some neighborhood of z_0 is contained in A and thus also in $A \cup B$. Similar if z_0 is from B then some neighborhood of z_0 is contained in B and thus in $A \cup B$. It implies that z_0 is an interior point of $A \cup B$. Hence the set $A \cup B$ is open.

To show that $A \cup B$ is connected we need to show that any two points z_1 and z_2 of $A \cup B$ can be connected by a polygonal line. Let the point z_3 is some point from the non-empty intersection of A and B. Then z_3 and z_1 are from the same set A or B. Therefore they can be connected by a polygonal line l_1. Also z_3 and z_2 are from the same set A or B. And they can be connected by a polygonal line l_2. Now it is obvious that z_1 and z_2 are connected by the polygonal line $l_1 \cup l_2$ passing through z_3. Hence $A \cup B$ is connected.

Since $A \cup B$ is simultaneously open and connected it is a region.

Solutions to Exercises 2.2

1. Using properties of limits and the fact that $\lim_{z\to i} z = i$ we have

$$\begin{aligned}
\lim_{z\to i} 3z^2 + 2z - 1 &= 3\lim_{z\to i} z^2 + 2\lim_{z\to i} z - 1 \\
&= 3\left(\lim_{z\to i} z\right)^2 + 2\lim_{z\to i} z - 1 \\
&= 3i^2 + 2i - 1 = -3 + 2i - 1 = -4 + 2i.
\end{aligned}$$

5. First we add to fractions under the sign of limit. We use the identity $z^2 + 1 = (z-i)(z+i)$. We have

$$\begin{aligned}
\frac{1}{z-i} - \frac{1}{z^2+1} &= \frac{z+i}{(z-i)(z+i)} - \frac{1}{(z-i)(z+i)} \\
&= \frac{z+i-1}{(z-i)(z+i)}.
\end{aligned}$$

We again use the properties from the Theorem 2.2.7 and the fact that $\lim_{z\to i} z = i$ to get

$$\begin{aligned}
\lim_{z\to i} \frac{1}{z-i} - \frac{1}{z^2+1} &= \lim_{z\to i} \frac{z+i-1}{(z-i)(z+i)} \\
&= \frac{\lim_{z\to i}(z+i-1)}{\lim_{z\to i} z - i \lim_{z\to i}(z+i)} \\
&= \frac{2i-1}{(\lim_{z\to i} z - i)2i} \\
&\quad [\text{since } \lim_{z\to c} f(z) = 0 \quad \Leftrightarrow \quad \lim_{z\to c} \frac{1}{f(z)} = \infty] \\
&= \frac{2i-1}{2i}\infty = \infty.
\end{aligned}$$

9. We evaluate using properties of limits and the fact that $\operatorname{Arg} z$ is always real to get

$$\begin{aligned}
\lim_{z\to -3} (\operatorname{Arg} z)^2 &= \lim_{z\to -3} |\operatorname{Arg} z|^2 \\
&= \left(\lim_{z\to -3} |\operatorname{Arg} z|\right)^2.
\end{aligned}$$

We know that in general $\operatorname{Arg} z$ is discontinuous on the ray $(-\infty, 0]$. But it turns out that the function $f(z) = |\operatorname{Arg} z|$ is continuous on the open ray $(-\infty, 0)$. Really if z_0 is from the open ray $(-\infty, 0)$ and z approaches to z_0 from the second quadrant then $\operatorname{Arg} z$ approaches to π. Therefore $|\operatorname{Arg} z|$ also approaches π. Now if z approaches to z_0 from the third quadrant then $\operatorname{Arg} z$ approaches to $-\pi$. But $|\operatorname{Arg} z|$ approaches π again. So, in either case $|\operatorname{Arg} z|$ approaches π. Therefore we conclude

$$\lim_{z\to z_0} |\operatorname{Arg} z| = \pi.$$

If we take $z_0 = -3$ we have

$$\begin{aligned}
\lim_{z\to z_0} (\operatorname{Arg} z)^2 &= \left(\lim_{z\to -3} |\operatorname{Arg} z|\right)^2 \\
&= \pi^2.
\end{aligned}$$

13. Since z approaches ∞ (thus $z \neq 0$) we can divide by z both the numerator and the denominator. We have

$$\lim_{z\to\infty} \frac{z+1}{3i\,z+2} \;=\; \lim_{z\to\infty} \frac{1+\frac{1}{z}}{3i+\frac{2}{z}} = \frac{1+\lim_{z\to\infty}\frac{1}{z}}{3i+\lim_{z\to\infty}\frac{2}{z}} = \frac{1}{3i} = \frac{i}{3i^2} = -\frac{i}{3}.$$

17. We have

$$\lim_{z\to 1} \frac{-1}{(z-1)^2} \;=\; -\left(\lim_{z\to 1}\frac{1}{z-1}\right)^2.$$

Now we notice that if $f(z)$ approaches 0 if z approaches 1 then $g(z) = \frac{1}{f(z)}$ approaches ∞. Also if $\lim_{z\to 1} g(z) = \infty$ then $\lim_{z\to 1} g(z)^2 = \infty$. Using these properties we compute

$$\left(\lim_{z\to 1}\frac{1}{z-1}\right)^2 \;=\; \infty.$$

Since multiplying by a non-zero constant does not change approaching to ∞ we have

$$\lim_{z\to 1} \frac{-1}{(z-1)^2} \;=\; \infty.$$

21. We know that a real exponential function $f(x) = e^x$ has the properties

$$\lim_{x\to+\infty} f(x) = 0,$$

and

$$\lim_{x\to-\infty} f(x) = +\infty.$$

So, we see that even for real z the function $f(z)$ has different limits in the positive and negative directions. Hence, it also cannot have a unique limit when z approaches infinity in the complex plane $\mathbb{C}$.

25. Approaching $z_0 = 0$ from the positive x-axis and the negative x-axis, respectively, thus

$$\lim_{\substack{y=0\\ x\to 0+}} \frac{z}{|z|} = \lim_{x\to 0+}\frac{x}{|x|} = 1,$$
$$\lim_{\substack{y=0\\ x\to 0-}} \frac{z}{|z|} = \lim_{x\to 0-}\frac{x}{|x|} = -1.$$

Since $\lim\limits_{\substack{y=0\\ x\to 0+}} \frac{z}{|z|} \neq \lim\limits_{\substack{y=0\\ x\to 0-}} \frac{z}{|z|}$, therefore the limit $\lim\limits_{z\to 0}\frac{z}{|z|}$ does not exist.

29. We need to show that

$$(1) \qquad \lim_{z\to\infty} f(z) = L \quad \Leftrightarrow \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = L;$$

Suppose the left statement is true. This means that for any $\varepsilon > 0$ there is an $R > 0$ such that $|z| > R$ it follows $|f(z) - L| < \varepsilon$. Now let $\varepsilon > 0$ be any. Let us denote $\delta = \frac{1}{R}$ where R is chosen as above. Then it follows that for any $|z| < \delta$ we have $\left|\frac{1}{z}\right| > R$. Thus it follows $|f(\frac{1}{z}) - L| < \varepsilon$. Hence $\lim_{z\to 0} f\left(\frac{1}{z}\right) = L$.

Now we suppose that the right statement is true. Again we pick any $\varepsilon > 0$. We can choose $\delta > 0$ such that if $|z| < \delta$ then $|f(\frac{1}{z}) - L| < \varepsilon$. Let us denote $R = \frac{1}{\delta}$. If $|z| > R$ then $|\frac{1}{z}| < \frac{1}{R} = \delta$. So,

$$|f(z) - L| \quad = \quad \left| f\left(\frac{1}{\frac{1}{z}}\right) - L \right| < \varepsilon.$$

Hence $\lim_{z\to\infty} f(z) = L$.

33. If $z \neq -1 - 3i$ then the function $h(z) = \frac{z-i}{z+1+3i}$ is continuous in z. The discontinuity at $z_0 = -1 - 3i$ is not removable because $z_0 - i = -1 - 3i - i = -1 - 4i$ and $z_0 + 1 + 3i = 0$. And we have

$$\begin{aligned} \lim_{z\to -1-3i} \frac{z-i}{z+1+3i} &= (-1-4i) \lim_{z\to -1-3i} \frac{1}{z+1+3i} \\ &= (-1-4i)\infty = \infty. \end{aligned}$$

37. We use the definition of $\sin z$ using the exponential function. We have

$$\sin z \quad = \quad \frac{e^{iz} - e^{-iz}}{2i}.$$

From the Example 2.2.19 we know that e^z is continuous everywhere in $\mathbb{C}$. Since $e^{iz} = f(g(z))$ for $g(z) = iz$ and $f(z) = e^{iz}$ and both functions are continuous it follows from Theorem 2.2.13 that e^{iz} is also continuous. Similar we show that the function e^{-iz} is continuous. Then by Theorem 2.2.13

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}e^{iz} - \frac{1}{2i}e^{-iz}$$

is continuous in $\mathbb{C}$.

41. (a) First, assume that f is continuous and A is open. Let z_0 be from $f^{-1}[A]$. This means that $f(z_0)$ is in A. Since A is open there is $\varepsilon > 0$ such that the ε-neighborhood is a subset of A, or $B_\varepsilon(f(z_0)) \subset A$. Since f is continuous at z_0 there is a $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$. Thus $f(z)$ is in A. And therefore z which is in $f^{-1}[f(z)]$ is in $f^{-1}[A]$. Therefore the whole δ-neighborhood of z_0 is in $f^{-1}[A]$. And it follows that $f^{-1}[A]$ is open.

Second, assume that $f^{-1}[A]$ is open whenever A is open. Let z_0 be any complex number and $\varepsilon > 0$ be any too. Then by assumption the ε-neighborhood $B_\varepsilon(f(z_0))$ of $f(z_0)$ is open too. Therefore $f^{-1}[B_\varepsilon(f(z_0))]$ is open. In particular, there is $\delta > 0$ such that $B_\delta(z_0) \subset f^{-1}[B_\varepsilon(f(z_0))]$. It follows that for any z such that $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \varepsilon$. Hence f is continuous.

(b) First, assume that $f^{-1}[A]$ is closed whenever A is closed. To show that f is continuous by part(a) it is enough to show that $f^{-1}[B]$ is open whenever B is open. So, let open B be given. Since the complement of an open set is closed (problem 17 Section 2.1) we have $A = \mathbb{C} \setminus B$ is closed. Therefore, by assumption $f^{-1}[A]$ is closed too. Now we use the identity

$$f^{-1}[B] = \mathbb{C} \setminus f^{-1}[A]$$

which is true for any set B and $A = \mathbb{C} \setminus B$. Really, if z is in $f^{-1}[B]$. Then $f(z)$ is in $B = \mathbb{C} \setminus A$. Therefore $f(z)$ is in the complement of $f^{-1}[A]$. On the other hand if z is in $\mathbb{C} \setminus f^{-1}[A]$ then $f(z)$ cannot be in $A = \mathbb{C} \setminus B$. And, thus $f(z)$ is in B. So, $f^{-1}(z)$ is in $f^{-1}[B]$. Since $f^{-1}[A]$ is closed it follows from (2) and Problem 17 Section 2.1 that the complement set $f^{-1}[B]$ is open. Thus f is continuous.

Second, assume that f is continuous and B is closed. Then $A = \mathbb{C} \setminus B$ is open. Therefore by part (a) it follows that $f^{-1}[A]$ is open. By (2) it follows that $f^{-1}[B] = \mathbb{C} \setminus f^{-1}[A]$. $f^{-1}[A]$ is open. Then by Problem 17 Section 2.1 it follows that its complement, $f^{-1}[B]$ is closed.

Solutions to Exercises 2.3

1. If we take $g(z) = 3z^2 + 2z$ and $h(z) = z - 1$ then we have

$$f(z) = 3(z-1)^2 + 2(z-1) = g(h(z)).$$

By the chain rule and the formula that $(z^n)' = nz^{n-1}$ for a positive integer n we have

$$f'(z) \quad = \quad g'(h(z))h'(z) = (6(z-1)+2)1 = 6(z-1)+2 = 6z-4.$$

5. By the quotient rule and the formula for the derivative of a polynomial it follows

$$\left(\frac{1}{z^3+1}\right)' \quad = \quad \frac{(1)'(z^3+1) - 1(z^3+1)'}{(z^3+1)^2} = \frac{0 - 3z^2}{(z^3+1)^2} = -\frac{3z^2}{(z^3+1)^2}.$$

9. We have $f(z) = z^{2/3} = (z^{1/3})^2$. So, if we take $g(z) = z^{1/3}$ and $h(z) = z^2$ then $f(z) = h(g(z))$. So, by the chain rule and the formulas for the derivatives of polynomial and n-th root we get

$$f'(z) \quad = \quad h'(g(z))g'(z) = 2g(z)\frac{1}{3}z^{(1-3)/3} = 2z^{1/3}\frac{1}{3}z^{-2/3} = \frac{2}{3}z^{-1/3}.$$

13. If we take $f(z) = z^{100}$ and $z_0 = 1$ then the limit is the definition of the derivative of f at z_0 since $f(z_0) = 1^{100} = 1$. By the formula for the derivative of z^n for a positive integer n we have

$$\lim_{z \to 1} \frac{z^{100} - 1}{z - 1} \quad = \quad f'(z_0) = 100 z_0^{100-1} = 100 \cdot 1^{99} = 100.$$

17. If z_0 is from the region $\{z: \ |z| < 1\}$ then the function f coincides with the function $g(z) = z$ on some neighborhood of z_0. Since $g(z)$ is differentiable everywhere then the limit used in the definition of differentiability of f coincides with the corresponding limit for g which we know is equal to $g'(z_0) = 1$. So, inside the region $\{z: \ |z| < 1\}$ we have $f'(z) = 1$.

Now if z_0 is from the region $\{z: \ |z| > 1\}$ then the function f coincides with the function $h(z) = z^2$ on some neighborhood of z_0. So, using the same argument as above we get that inside this region $\{z: \ |z| > 1\}$ we have $f'(z) = (z^2)' = 2z$.

The question of differentiability remains for z_0 from the circle $\{z: \ |z| = 1\}$. We have $f(z_0) = z_0$. So, if $z_0 \neq 1$ then $z_0 \neq z_0^2$ and since f coincides with $h(z) = z^2$ inside the region $\{z: \ |z| > 1\}$. And $h(z)$ is continuous everywhere. So, if z approaches z_0 inside the region $\{z: \ |z| > 1\}$ then $f(z) = h(z)$ approaches $h(z_0) = z_0^2$ which we know is not equal to $f(z_0)$. So, f is not continuous at a such z_0. Now if $z_0 = 1$ then both g and h coincide at this point. Now let C_1 be any path which approaches $z_0 = 1$ inside the region $\{z: \ |z| < 1\}$ (for example $C_1 = \{z: \ z = r, \ 0 < r < 1\}$) and C_2 be any path which approaches $z_0 = 1$ inside the region $\{z: \ |z| > 1\}$ (for example $C_2 = \{z: \ z = r, \ r > 1\}$). Since g and h are differentiable everywhere we have

$$\begin{aligned} \lim_{\substack{z \to z_0 \\ z \text{ on } C_1}} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\substack{z \to z_0 \\ z \text{ on } C_1}} \frac{g(z) - g(z_0)}{z - z_0} \\ &= g'(z_0) = 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{\substack{z \to z_0 \\ z \text{ on } C_2}} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{\substack{z \to z_0 \\ z \text{ on } C_2}} \frac{h(z) - h(z_0)}{z - z_0} \\ &= h'(z_0) = 2z_0 = 2. \end{aligned}$$

Since we have different limits when we approach $z_0 = 1$ from different directions, it follows that the derivative of $f(z)$ does not exist at $z_0 = 1$. Finally, we see that the derivative of f exist everywhere except the circle $\{z : \ |z| = 1 \}$ and

$$f'(z) = \begin{cases} 1 & \text{if } |z| < 1, \\ 2z & \text{if } |z| > 1, \end{cases}$$

21. We take $g(z) = z^{p/q}$ and $f(z) = z^q$ in Theorem 4 and get $h(z) = f(g(z)) = (z^{p/q})^q = z^p$ there. Since $f(z) \neq 0$ for any $z \neq 0$ and $z_0^{p/q} \neq 0$ for $z_0 \neq 0$ and not on the negative real axis we have $f'(g(z_0)) \neq 0$. We also know that

$$g(z) = e^{\frac{p}{q}(\operatorname{Log}(z) + i2k\pi)}$$

for some integer k. So, g is a composition of two continuous functions on its domain. Therefore g is also continuous. Now we can use Theorem 4 to get

$$\begin{aligned} \frac{d}{dz} z^{p/q} &= \frac{h'(z)}{f'(g(z))} = \frac{pz^{p-1}}{qg(z)^{q-1}} = \frac{pz^{p-1}}{qz^{(p/q)(q-1)}} = \frac{pz^p z^{-1}}{qz^p z^{-p/q}} \\ & \quad \text{[if we divide the numerator and the denominator by } z^p \\ & \quad \quad \text{and multiply them by } z^{p/q} z] \\ &= \frac{p}{qz} z^{p/q}. \end{aligned}$$

25. (a) Since $(i^2 + 1)^7 = i^6 + 1 = 0$, we have

$$\lim_{z \to i} \frac{\left(z^2 + 1\right)^7}{z^6 + 1} = \frac{7\left(z^2 + 1\right)^6 (2z)}{6z^5} \bigg|_{z=i} = 0.$$

(b) We have

$$\lim_{z \to i} \frac{z^3 + (1 - 3i)z^2 + (i - 3)z + 2 + i}{z - i} = \frac{3z^2 + 2(1 - 3i)z + i - 3}{1} \bigg|_{z=i} = 3i.$$

Solutions to Exercises 2.4

1. If $(x,y) \neq (0,0)$ then we can compute the derivatives directly since $x^2+y^2 \neq 0$. We have

$$\begin{aligned} u_x &= \frac{(xy)_x(x^2+y^2) - xy(x^2+y_2)_x}{(x^2+y^2)^2} \\ &= \frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2}. \end{aligned}$$

Similar

$$\begin{aligned} u_y &= \frac{(xy)_y(x^2+y^2) - xy(x^2+y_2)_y}{(x^2+y^2)^2} \\ &= \frac{x(x^2+y^2) - 2xy^2}{(x^2+y^2)^2}. \end{aligned}$$

Now if $(x,y) = (0,0)$ then we can notice that for any $x_1 \neq 0$ we have

$$u(x_1, 0) = \frac{x_1 0}{x_1^2 + 0^2} = 0.$$

Therefore

$$u_x(0,0) = \lim_{x_1 \to 0} \frac{u(x_1,0) - u(0,0)}{x_1 - 0} = \lim_{x_1 \to 0} \frac{0}{x_1} = 0.$$

Therefore the partial derivative of u with respect to x exists at $(0,0)$ and is equal to 0.

Similarly, we notice that for any $y_1 \neq 0$ we have

$$u(0, y_1) = \frac{0 y_1}{0^2 + y^2} = 0.$$

Therefore

$$u_y(0,0) = \lim_{y_1 \to 0} \frac{u(0,y_1) - u(0,0)}{y_1 - 0} = \lim_{y_1 \to 0} \frac{0}{y_1} = 0.$$

Hence u_y exists at $(0,0)$ and is equal to 0.

To show that $u(x,y)$ is not continuous at $(0,0)$ we pick a point (x,x) for any number $x \neq 0$. We compute

$$u(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}.$$

Therefore if x approaches zero then (x,x) approaches $(0,0)$ but $u(x,x) = 1/2$ does not approach $u(0,0) = 0$. Therefore u is discontinuous at $(0,0)$.

5. Since f and g are differentiable at $x = x_0$ and $y = y_0$ correspondingly we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon_1(x)|x - x_0|$$

and

$$g(y) = g(y_0) + g'(y_0)(y - y_0) + \varepsilon_2(y)|y - y_0|$$

where $\varepsilon_1(x) \to 0$ as $x \to x_0$ and $\varepsilon_2(y) \to 0$ as $y \to y_0$. Therefore we have

$$\begin{aligned} u(x,y) &= f(x)g(y) \\ &= (f(x_0) + f'(x_0)(x - x_0) + \varepsilon_1(x)|x - x_0|)(g(y_0) + g'(y_0)(y - y_0) \\ &\quad + \varepsilon_2(y)|y - y_0|) \\ &= f(x_0)g(y_0) + g(y_0)f'(x_0)(x - x_0) + f(y_0)g'(x_0) + m(x,y), \end{aligned}$$

where

$$\begin{aligned} m(x,y) &= f'(x_0)g'(y_0)(x-x_0)(y-y_0) \\ &\quad +\varepsilon_1(x)|x-x_0|(g(y_0)+g'(y_0)(y-y_0)+\varepsilon_2(y)|y-y_0|) \\ &\quad +\varepsilon_2(y)|y-y_0|(f(x_0)+f'(x_0)(x-x_0)). \end{aligned}$$

To show that that u is differentiable at $z_0 = (x_0, y_0)$ it is enough to check that

$$\lim_{z\to z_0} \frac{m(z)}{|z-z_0|} = 0.$$

First, we show that $\lim_{z\to z_0} \frac{(x-x_0)(y-y_0)}{|z-z_0|} = 0$. To see it we will use the squeeze theorem and the inequality $ab \le \frac{a^2+b^2}{2}$. To prove the inequality we start from the obvious inequality $(a-b)^2 \ge 0$ which is equivalent to $a^2 - 2ab + b^2 \ge 0$ or if we add $2ab$ to both sides we get $a^2+b^2 \ge 2ab$ and if we divide by 2 we get $\frac{a^2+b^2}{2} \ge ab$ which is the needed inequality. Now for $a = x - x_0$ and $b = y - y_0$ we get

$$(x-x_0)(y-y_0) \le \frac{(x-x_0)^2+(y-y_0)^2}{2} = \frac{|z-z_0|^2}{2} \quad \text{by Pythagorian theorem.}$$

We also can put $a = -(x-x_0)$ and $b = y - y_0$ to receive similar

$$-(x-x_0)(y-y_0) \le \frac{(x-x_0)^2+(y-y_0)^2}{2} = \frac{|z-z_0|^2}{2}.$$

If we multiply the whole inequality by (-1) then we need to reverse the inequality and we get

$$(x-x_0)(y-y_0) \ge -\frac{|z-z_0|^2}{2}.$$

If we combine the inequalities (1) and (1) we conclude that

$$-\frac{|z-z_0|^2}{2} \le (x-x_0)(y-y_0) \le \frac{|z-z_0|^2}{2}.$$

If we divide these inequalities by $|z-z_0|$ we get

$$-\frac{|z-z_0|}{2} \le \frac{(x-x_0)(y-y_0)}{|z-z_0|} \le \frac{|z-z_0|}{2}.$$

We have $\lim_{z\to z_0} -\frac{|z-z_0|}{2} = 0$ and $\lim_{z\to z_0} \frac{|z-z_0|}{2} = 0$. So, by squeeze theorem

$$\lim_{z\to z_0} \frac{(x-x_0)(y-y_0)}{|z-z_0|} = 0.$$

We notice that

$$\begin{aligned} &\lim_{z\to z_0} (g(y_0)+g'(y_0)(y-y_0)+\varepsilon_2(y)|y-y_0|) \\ = &\lim_{y\to y_0} (g(y_0)+g'(y_0)(y-y_0)+\varepsilon_2(y)|y-y_0|) = g(y_0). \end{aligned}$$

Now we find that

$$\begin{aligned} &\lim_{z\to z_0} \frac{\varepsilon_1(x)|x-x_0|(g(y_0)+g'(y_0)(y-y_0)+\varepsilon_2(y)|y-y_0|)}{|z-z_0|} \\ = &\lim_{z\to z_0} \frac{\varepsilon_1(x)|x-x_0|}{|z-z_0|} \lim_{z\to z_0} (g(y_0)+g'(y_0)(y-y_0)+\varepsilon_2(y)|y-y_0|) = 0g(y_0) = 0 \end{aligned}$$

since $\frac{|x-x_0|}{|z-z_0|}$ is bounded by 1 and $\varepsilon_1(x) \to 0$ if $z \to z_0$.

Similarly, we find that

$$\begin{aligned} & \lim_{z \to z_0} \frac{\varepsilon_2(y)|y-y_0|(f(x_0)+f'(x_0)(y-y_0))}{|z-z_0|} \\ = & \lim_{z \to z_0} \frac{\varepsilon_2(y)|y-y_0|}{|z-z_0|} \lim_{z \to z_0} (f(x_0)+f'(x_0)(x-x_0)) = 0f(x_0) = 0. \end{aligned}$$

Finally adding the limits (1)-(1) we conclude that

$$\lim_{z \to z_0} \frac{m(z)}{|z-z_0|} = 0.$$

9. Project Problem: Is it true that if $u_y(x,y) = 0$ for all (x,y) is a region Ω, then $u(x,y) = \phi(x)$; that is, u depends only on x? The answer is no in general, as the following counterexamples show.

For (x,y) in the region Ω shown in Figure 8, consider the function

$$u(x,y) = \begin{cases} 0 & \text{if } x > 0, \\ \text{sgn} y & \text{if } x \le 0, \end{cases}$$

where the **signum function** is defined by $\text{sgn} y = -1, 0, 1$, according as $y < 0, y = 0$, or $y > 0$. Show that $u_y(x,y) = 0$ for all (x,y) in Ω but that u is not a function of x alone.

Note that in the previous example u_x does not exist for $x = 0$. We now construct a function over the same region Ω for which the partial exist, $u_y = 0$, and u is not a function of x alone. Show that these properties hold for

$$u(x,y) = \begin{cases} 0 & \text{if } x > 0, \\ e^{-1/x^2} \text{sgn} y & \text{if } x \le 0. \end{cases}$$

Come up with a general condition on Ω that guarantees that whether $u_y = 0$ on Ω then u depends only on x. [Hint: Use the mean value theorem as applied to vertical line segments in Ω.]

For any point in the region $(x,y) \in \Omega$, the partial derivative $u_y(x,y)$ at this point can be denoted as

$$u_y(x,y) = \lim_{h \to 0} \frac{u(x,y+h) - u(x,y)}{h}.$$

By the definition of the function $u(x,y)$,

If $x > 0$, then $u_y(x,y) = 0$;

If $x < 0$ and $y < 0$ or $y > 0$, since the region does not contain the boundary, thus $u_y(x,y) = 0$;

Otherwise, consider the line segments along the y-axis. Since the region Ω does not contain the origin, for $y > 0, x = 0$, we have

$$u_y(0,y) = \lim_{h \to 0} \frac{u(0,y+h) - u(0,y)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0.$$

Similarly, for $y < 0, x = 0$, we have

$$u_y(x,y) = \lim_{h \to 0} \frac{u(0,y+h) - u(0,y)}{h} = \lim_{h \to 0} \frac{-1-(-1)}{h} = 0.$$

Hence, it implies that the partial derivative satisfies $u_y(x,y) = 0$ for all $(x,y) \in \Omega$, but $u(x,y)$ depends on both x and y.

Consider the u_x in the previous example. Since

$$u_x(0, y) = \lim_{h \to 0} \frac{u(0 + h, y) - u(0, y)}{h}.$$

If $y > 0$ (similar applying to $y < 0$), then

$$\lim_{h \to 0+} \frac{u(h, y) - u(0, y)}{h} = \lim_{h \to 0+} \frac{0 - 1}{h} \neq \lim_{h \to 0-} \frac{1 - 1}{h} = 0,$$

which implies that u_x does not exist for $x = 0$.

Now, consider the new function from this example. For the partial derivative $u_y(x, y)$,

If $x > 0$, then $u_y(x, y) = u_x(x, y) = 0$;

If $x < 0$, then we can also obtain that $u_y(x, y) = 0$, whenever $y > 0$ or $y < 0$ since $e^{1/x^2} > 0, \forall x \in \mathbb{R}\backslash\{0\}$. And for u_x, since e^{1/x^2} is differentiable at $\mathbb{R}\backslash\{0\}$, thus u_x also exists.

But u is not a function of x alone. Therefore, the properties hold for this new $u(x, y)$. Assume that the region Ω is convex or path-connected.

Fix any $x = x_0$, such that we can obtain a vertical line segment in Ω. Then for any $(x_0, y_1), (x_0, y_2) \in \Omega$, since the region Ω is convex, then the vertical line segment joins (x_0, y_1) and (x_0, y_2) is contained in Ω. By mean value theorem, we have

$$u(x_0, y_1) - u(x_0, y_2) = u_y(x_0, \xi)(y_1 - y_2),$$

for some ξ between y_1 and y_2. Since $u_y = 0$ on Ω, then $u(x_0, y_1) = u(x_0, y_2)$. Since x_0 and then $(x_0, y_1), (x_0, y_2) \in \Omega$ are all arbitrarily chosen, thus u depends only on x.

Solutions to Exercises 2.5

1. For $z = x + iy$ we get $z = u(x,y) + iv(x,y)$ for $u(x,y) = x$ and $v(x,y) = y$. Differentiating u with respect to x and y, we find

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 0.$$

Differentiating v with respect to x and y, we find

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 1.$$

Comparing these derivatives we see clearly that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Hence the Cauchy-Riemann equations are satisfied at all points. The partial derivatives are clearly continuous everywhere, so by Theorem 2.4.4, $u(x,y)$ and $v(x,y)$ are differentiable everywhere.. Appealing to Theorem 2.5.1, we conclude that z is analytic at all points, or, entire. We compute the derivative as $f'(x+iy) = u_x(x,y) + iv_x(x,y)$, giving

$$f'(z) = 1 + i0 = 1.$$

5. Let $f(z) = e^{\overline{z}}$, $z = x + iy$. Then

$$\begin{aligned} e^{\overline{z}} &= e^{x-iy} \\ &= e^x(\cos(-y) + i\,\sin(-y)) \\ &= e^x\cos(y) - i\,e^x\sin(y) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

for $u(x,y) = e^x\cos(y)$ and $v(x,y) = -e^x\sin(y)$. Differentiating, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x\cos(y), & \frac{\partial u}{\partial y} &= -e^x\sin(y), \\ \frac{\partial v}{\partial x} &= -e^x\sin(y), & \frac{\partial v}{\partial y} &= -e^x\cos(y). \end{aligned}$$

If the Cauchy-Riemann equations are to be satisfied, we must have $e^x\cos(y) = 0$ and $e^x\sin(y) = 0$. However, this implies that $\sin(y) = \cos(y) = 0$, and sin and cos are never simultaneously zero. Thus, the Cauchy-Riemann equations cannot be satisfied at any point, and $f(z)$ is nowhere analytic.

9. For $z = x + iy$ we get

$$\begin{aligned} ze^z &= (x+iy)e^{x+iy} \\ &= (x+iy)e^x(\cos y + i\,\sin y) \\ &= e^x(x\cos y - y\sin y + i(y\cos y + x\sin y)) \\ &= u(x,y) + iv(x,y) \end{aligned}$$

for $u(x,y) = e^x(x\cos y - y\sin y)$ and $v(x,y) = e^x(y\cos y + x\sin y)$. Differentiating u using the product rule with respect to x, we find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(e^x x\cos y - e^x y\sin y\right) \\ &= \left(\frac{\partial}{\partial x}(e^x)x + e^x\frac{\partial}{\partial x}(x)\right)\cos y - \frac{\partial}{\partial x}(e^x)y\sin y \\ &= (e^x x + e^x)\cos y - e^x y\sin y \\ &= e^x(x\cos y + \cos y - y\sin y). \end{aligned}$$

Also differentiating u with respect to y, we find

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(e^x x\cos y - e^x y\sin y\right)\\ &= e^x x\frac{\partial}{\partial y}(\cos y) - e^x\left(\frac{\partial}{\partial y}(y)\sin y + y\frac{\partial}{\partial y}(\sin y)\right)\\ &= -e^x x\sin y - e^x\left(\sin y + y\cos y\right)\\ &= e^x(-x\sin y - \sin y - y\cos y)\end{aligned}$$

Similarly, we compute the derivatives v with respect to x and y

$$\begin{aligned}\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\left(e^x y\cos y + e^x x\sin y\right)\\ &= \frac{\partial}{\partial x}(e^x)y\cos y + \left(\frac{\partial}{\partial x}(e^x)x + e^x\frac{\partial}{\partial x}(x)\right)\sin y\\ &= e^x y\cos y + (e^x x + e^x)\sin y\\ &= e^x(y\cos y + \sin y + x\sin y)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(e^x y\cos y + e^x x\sin y\right)\\ &= e^x\left(\frac{\partial}{\partial y}(y)\cos y + y\frac{\partial}{\partial y}(\cos y) + x\frac{\partial}{\partial y}(\sin y)\right)\\ &= e^x\left(\cos y - y\sin y + x\cos y\right).\end{aligned}$$

Comparing these derivatives we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Hence the Cauchy-Riemann equations are satisfied at all points. Appealing to Theorem 2.5.1, we conclude that ze^z is analytic at all points, or entire. We compute the derivative as $f'(z) = u_x(x,y) + iv_x(x,y)$, getting

$$\frac{d}{dz}ze^z = e^x(x\cos y + \cos y - y\sin y) + ie^x(-x\sin y - \sin y - y\cos y).$$

13. Let $z = x+iy$. This implies $f(z) = |z|^2 = x^2+y^2$. Hence, $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. Therefore,

$$u_x(x,y) = 2x, u_y(x,y) = 2y, \text{ and } v_x(x,y) = v_y(x,y) = 0.$$

The fact that $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$ implies $x = 0$ and $y = 0$. Since the function $f(z) = |z|^2$ is not differentiable in any neighborhood of $z = 0$, it cannot be analytic even at the point $z = 0$.

17. From the Example 2.5.4, $\sin z$ is entire and $\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z$. By Exercise 2.5.10, $\cos z$ is entire and $\frac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z$. By Theorem 2.3.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z\cos z = \cos z\frac{\mathrm{d}}{\mathrm{d}z}\sin z + \sin z\frac{\mathrm{d}}{\mathrm{d}z}\cos z = \cos^2 z - \sin^2 z.$$

Since $\sin z$ and $\cos z$ are entire, $\sin z\cos z$ is also entire.

21. By Exercise 2.5.12, $\cosh z$ is entire and $\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z$. Applying Theorem 2.3.11, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\cosh\bigl(z^2+3i\bigr) = \sinh\bigl(z^2+3i\bigr)\frac{\mathrm{d}}{\mathrm{d}z}\bigl(z^2+3i\bigr) = 2z\sinh\bigl(z^2+3i\bigr).$$

Since $\cosh z$ and z^2 are entire, $\cosh\bigl(z^2+3i\bigr)$ is also entire.

25. By (2.5.13) and Theorem 2.3.11, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}{}^{-1/2} = -\frac{1}{2}(z-i)^{-3/2}\frac{\mathrm{d}}{\mathrm{d}z} = -\frac{1}{2(z-i)^{3/2}},$$

and this is analytic for all $z-i \in \mathbb{C}\setminus(-\infty,0]$, that is, for all $z \in \mathbb{C}\setminus\{z : \operatorname{Re} z \le 0,\ \operatorname{Im} z = i\}$.

29. By L'Hospital's rule,

$$\lim_{z\to 0}\frac{\operatorname{Log}(z+1)}{z} = \frac{\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Log}(z+1)}{\frac{\mathrm{d}}{\mathrm{d}z}z}\Bigg|_{z=0} = \frac{1}{z+1}\Bigg|_{z=0} = 1.$$

33. Since $f = u + iv$ is analytic in a region Ω, then for any $(x,y) \in \Omega$, we have

$$u_x(x,y) = v_y(x,y), \text{ and } u_y(x,y) = -v_x(x,y).$$

We have two cases:

(1) If $\operatorname{Re} f$ is constant on Ω, then $u_x(x,y) = u_y(x,y) = 0$ on Ω which implies that $v_x(x,y) = v_y(x,y) = 0$ on Ω. Therefore, f is constant on Ω.

(2) If $\operatorname{Im} f$ is constant on Ω, then $v_x(x,y) = v_y(x,y) = 0$ on Ω which implies that $u_x(x,y) = u_y(x,y) = 0$ on Ω. Therefore, f is constant on Ω.

Hence, f is constant on Ω if either $\operatorname{Re} f$ or $\operatorname{Im} f$ are constant on Ω.

37. We want to rotate the line to the subset of real axis and use Exercise 33. In order to rotate the line, we first choose a complex number $c \in f[\Omega]$. Then $f[\Omega] - c$ is a subset of a line passing through the origin. We may then choose an angle θ to complete the rotation so that

$$g(z) = e^{i\theta}(f(z) - c)$$

is a real-valued function on Ω. Since c and θ are constant, $g(z)$ is also analytic with $\operatorname{Im} f \equiv 0$. Therefore, by Exercise 33, $g(z)$ is constant in Ω, and $f(z) = e^{-i\theta}g(z) + c$ is constant in Ω.

41. Letting $f(z) = z^n = r^n(\cos(n\theta) + i\sin(n\theta))$, we have that $u(r,\theta) = r^n\cos(n\theta)$ and $v(r,\theta) = r^n\sin(n\theta)$. Thus, we have

$$\begin{aligned}
\frac{\partial u}{\partial r} &= nr^{n-1}\cos(n\theta), & \frac{\partial u}{\partial \theta} &= -nr^n\sin(n\theta),\\
\frac{\partial v}{\partial r} &= nr^{n-1}\sin(n\theta), & \frac{\partial v}{\partial \theta} &= nr^n\cos(n\theta).
\end{aligned}$$

We can see that the polar Cauchy-Riemann equations hold everywhere for $n > 0$ and for $z \neq 0$ for $n < 0$, and these partial derivatives are continuous where they are defined, so $f(z)$ is analytic on $\mathbb{C}$ for $n > 0$ and on $\mathbb{C}\setminus\{0\}$ for $n < 0$, and we have that

$$\begin{aligned}
f'(z) &= e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\bigl(nr^{n-1}\cos(n\theta) + inr^{n-1}\sin(n\theta)\bigr)\\
&= e^{-i\theta}nr^{n-1}e^{in\theta} = nr^{n-1}e^{i(n-1)\theta} = nz^{n-1}.
\end{aligned}$$

Solutions to Exercises 3.1

1. Given $z_1 = 1+i$ and $z_2 = -1-2i$, apply (3.1.2) to obtain the parametrization of the line segment $[z_1,\ z_2]$:

$$\gamma(t) = (1-t)z_1 + tz_2 = (1-t)(1+i) + t(-1-2i), \qquad 0 \le t \le 1,$$

or

$$\gamma(t) = t(-2-3i) + 1 + i, \qquad 0 \le t \le 1.$$

5. The parametrization is given by $\gamma(t) = e^{it}$ for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$.

9. The parametrization is given by $\gamma(t) = -3 + 2i + 5e^{it}$ for $-\frac{\pi}{2} \le t \le 0$.

13. From Exercise 10, we have that

$$\gamma(t) = \begin{cases} 5i - 3it & 0 \le t \le 1 \\ 2e^{\frac{i\pi t}{2}} & 1 \le t \le 4. \end{cases}$$

By (3.1.3), we have

$$\gamma^*(t) = \gamma(b+a-t) = \begin{cases} 2e^{\frac{i\pi}{2}(4-t)} & 0 \le t \le 3 \\ 5i - 3i(4-t) & 3 \le t \le 4. \end{cases}$$

17. This is a negatively-oriented circle with center $-i$ and radius $\frac{1}{2}$.

21. By the chain rule, we have

$$\frac{d}{dt}(2+i)\cos(3it) = (2+i)\frac{d}{dt}\cos(3it) = -3i(2+i)\sin(3it) = (3-6i)\sin(3it).$$

25. In complex form, we have

$$\begin{aligned} \gamma(t) = x(t) + iy(t) &= (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right) + i\left((a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)\right) \\ &= (a-b)(\cos t + i\sin t) + b\left(\cos\left(\frac{a-b}{b}t\right) - i\sin\left(\frac{a-b}{b}t\right)\right) \\ &= (a-b)e^{it} + be^{-i\frac{a-b}{b}t}. \end{aligned}$$

29. We first verify the endpoints.

$$\gamma(0) = 0, \gamma\left(\frac{1}{3}\right) = 1+i; \gamma\left(\frac{2}{3}\right) = -1+i; \gamma(1) = 0.$$

Note $\gamma_1(t) = 3t(1+i)$ for $0 \le t \le \frac{1}{3}$ represents the subset of the line $y = x$. Observe $\gamma_2(t) = 3 + i - 6t$ for $\frac{1}{3} \le t \le \frac{2}{3}$ represents the subset of the line $y = 1$. Notice $\gamma_3(t) = (-1+i)(3-3t)$ for $\frac{2}{3} \le t \le 1$ represents the subset of the line $y = -x$.

Solutions to Exercises 3.2

1. We have

$$\int_0^{2\pi} e^{3ix}\,dx = \frac{1}{3i}e^{3ix}\Big|_0^{2\pi} = \frac{1}{3i}(e^{6i\pi}-1) = 0.$$

5. Write

$$\begin{aligned}\frac{x+i}{x-i} &= \frac{x+i}{x-i}\left(\frac{x+i}{x+i}\right)\\ &= \frac{x^2+2ix-1+i}{x^2+1}\\ &= 1-\frac{2}{x^2+1}+\frac{2ix}{x^2+1}.\end{aligned}$$

So

$$\begin{aligned}\int_{-1}^{1}\frac{x+i}{x-i}\,dx &= \int_{-1}^{1}1\,dx - \int_{-1}^{1}\frac{2}{x^2+1}\,dx + \overbrace{2i\int_{-1}^{1}\frac{x}{x^2+1}\,dx}^{=0,\ \text{odd integrand}}\\ &= x-2\tan^{-1}x\Big|_{-1}^{1} = 2-\pi.\end{aligned}$$

9. Proceed as in Example 2:

$$f(x)=\begin{cases}(3+2i)x & \text{if } -1\le x\le 0,\\ ix^2 & \text{if } 0\le x\le 1;\end{cases}$$

hence an antiderivative

$$F(x)=\begin{cases}\frac{3+2i}{2}x^2+C & \text{if } -1\le x\le 0,\\ \frac{i}{3}x^3 & \text{if } 0\le x\le 1.\end{cases}$$

Setting $F(0+)=F(0-)$, we obtain $0=\frac{3+2i}{2}0+C$ or $C=0$. Hence a continuous antiderivative of f is

$$F(x)=\begin{cases}\frac{3+2i}{2}x^2 & \text{if } -1\le x\le 0,\\ \frac{i}{3}x^3 & \text{if } 0\le x\le 1.\end{cases}$$

By Theorem 3.2.7, we have that

$$\int_{-1}^{1} f(x)\,dx = F(1)-F(-1) = \frac{i}{3}-\frac{3+2i}{2} = -\frac{3}{2}-\frac{2}{3}i.$$

13. Parameterize $C_1(0)$ by $\gamma(t)=e^{it}$ where $0\le t\le 2\pi$. It follows that $\gamma'(t)=ie^{it}$. Then,

$$\int_{C_1(0)}(2z+i)\,dz = \int_0^{2\pi}(2e^{it}+i)ie^{it}\,dt = 0.$$

17. For the given $\gamma(t)$, we have $\gamma'(t) = ie^{it} - 2ie^{-it}$. Thus, the integral becomes

$$\int_\gamma (z + 2\overline{z})\, dz = \int_0^{2\pi} \left(e^{it} + 2e^{-it} + 2\overline{e^{it} + 2e^{-it}}\right)\left(ie^{it} - 2ie^{-it}\right)\, dt$$

$$= \int_0^{2\pi} \left(e^{it} + 2e^{-it}\right) d\left(e^{it} + 2e^{-it}\right) + 2\int_0^{2\pi} \overline{e^{it} + 2e^{-it}}\left(ie^{it} - 2ie^{-it}\right)\, dt$$

$$= \left[\frac{\left(e^{it} + 2e^{-it}\right)^2}{2}\right]_0^{2\pi} + 2\int_0^{2\pi} \left(e^{-it} + 2e^{it}\right)\left(ie^{it} - 2ie^{-it}\right)\, dt$$

$$= 2i\int_0^{2\pi} \left(2e^{2it} - 2e^{-2it} - 3\right)\, dt = 2i\int_0^{2\pi} \left(4\sin(2t) - 3\right)\, dt$$

$$= 2i[-2\cos(2t) - 3t]_0^{2\pi} = -12\pi i.$$

21. Write

$$\begin{aligned} I &= \int_{[z_1,\, z_2,\, z_3,\, z_4,\, z_1]} z\, dz \\ &= \int_{[z_1,\, z_2]} z\, dz + \int_{[z_2,\, z_3]} z\, dz + \int_{[z_3,\, z_4]} z\, dz + \int_{[z_4,\, z_1]} z\, dz \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where $z_1 = 0$, $z_2 = 1$, $z_3 = 1 + i$, and $z_4 = i$. To evaluate I_1, parametrize $[z_1,\ z_2]$ by $z = \gamma_1(t) = t$, $0 \le t \le 1$, $dz = dt$. So

$$I_1 = \int_{[z_1,\, z_2]} z\, dz = \int_0^1 t\, dt = \frac{1}{2}.$$

To evaluate I_2, parametrize $[z_2,\ z_3]$ by $z = \gamma_2(t) = 1 + it$, $0 \le t \le 1$, $dz = i\, dt$. So

$$I_2 = \int_{[z_2,\, z_3]} z\, dz = \int_0^1 (1 + it)i\, dt = i(t + \frac{i}{2}t^2)\Big|_0^1 = i(1 + \frac{i}{2}) = -\frac{1}{2} + i.$$

To evaluate I_3, parametrize $[z_3,\ z_4]$ by $z = \gamma_3(t) = (1 - t) + i$, $0 \le t \le 1$, $dz = -dt$. So

$$I_3 = \int_{[z_3,\, z_4]} z\, dz = -\int_0^1 ((1 - t) + i)\, dt = -(1 + i) + \frac{1}{2} = -\frac{1}{2} - i.$$

To evaluate I_4, parametrize $[z_4,\ z_1]$ by $z = \gamma_4(t) = (1 - t)i$, $0 \le t \le 1$, $dz = -idt$. So

$$I_4 = \int_{[z_4,\, z_1]} z\, dz = -i\int_0^1 i(1 - t)\, dt = 1 - \frac{1}{2} = \frac{1}{2}.$$

Finally, adding the four integrals, we obtain $I = 0$.

25. We apply the definition of the path integral, with $\gamma(t) = ae^{it} + be^{-i\frac{a}{2}t}$, $0 \le t \le 2\pi$, $\gamma'(t) = aie^{it} - i\frac{ab}{2}e^{-i\frac{a}{2}t}$, $a = 8$, $b = 5$:

$$\begin{aligned}\int_\gamma z\,dz &= \int_0^{2\pi} \left(ae^{it} + be^{-i\frac{a}{2}t}\right)\left(aie^{it} - i\frac{ab}{2}e^{-i\frac{a}{2}t}\right) dt \\ &= \int_0^{2\pi} \left(ia^2e^{2it} - i\frac{ab^2}{2}e^{-iat} + iabe^{i(1-\frac{a}{2})t} - i\frac{a^2b}{2}e^{i(1-\frac{a}{2})t}\right) dt = 0.\end{aligned}$$

29. We have $f(z) = z\overline{z}$, $\gamma(t) = (1-t)(2+i) + t(-1-i) = (-3-2i)t + 2 + i$, $0 \le t \le 1$, $dz = (-3-2i)dt$. So

$$\begin{aligned}\int_{[z_1,\,z_2]} (x^2+y^2)\,dz &= \int_0^1 ((-3-2i)t+2+i)((-3+2i)t+2-i)(-3-2i)dt \\ &= (-3-2i)\int_0^1 \big(13t^2 + (-3-2i)(2-i)t + (2+i)(-3+2i)t + 5\big)dt \\ &= (-3-2i)\left(\frac{13}{3} + \frac{(-3-2i)(2-i)}{2} + \frac{(2+i)(-3+2i)}{2} + 5\right) = -4 - \frac{8}{3}i.\end{aligned}$$

33. We have that $\gamma'(t) = ie^{it} = -\sin t + i\cos t$, so we have

$$\left|\gamma'(t)\right| = \sqrt{|-\sin t|^2 + |\cos t|^2} = 1.$$

Thus, the arc length is

$$\ell(\gamma) = \int_0^{\pi/6} 1\,dt = \frac{\pi}{6}.$$

37. Let $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$. It follows that $|z| = 2$. This implies that $|z-1| \ge |z| - |1| = 1$, so $\left|\frac{1}{z-1}\right| \le 1$. We also have that $\ell(\gamma) = 4\pi$. Thus, $\left|\int_{C_2(0)} \frac{1}{z-1}\,dz\right| \le 4\pi$.

41. (a) Suppose $m = n$. Then

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi.$$

Otherwise, with $m \ne n$, we have

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx = \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \left[\frac{1}{i(m-n)}e^{i(m-n)x}\right]_{-\pi}^{\pi} = 0.$$

(b) For all integers m and n, we have

$$\begin{aligned}e^{imx}e^{-inx} &= (\cos mx + i\sin mx)(\cos nx - i\sin nx) \\ &= \cos mx\cos nx + \sin mx\sin nx + i(\cos nx\sin mx - \cos mx\sin nx).\end{aligned}$$

Now suppose m and n are non-negative integers with $m \neq n$. Without loss of generality, we may assume $n \neq 0$. Then we also have $m \neq -n$ and

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{imx}e^{-inx} + e^{imx}e^{inx}}{2} \, dx$$
$$= \frac{1}{2} \operatorname{Re} \left(\int_{-\pi}^{\pi} e^{imx}e^{-inx} \, dx + \int_{-\pi}^{\pi} e^{imx}e^{inx} \, dx \right) = \frac{1}{2} \operatorname{Re} (0 + 0) = 0.$$

Similarly,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{imx}e^{-inx} - e^{imx}e^{inx}}{2} \, dx$$
$$= \frac{1}{2} \operatorname{Re} \left(\int_{-\pi}^{\pi} e^{imx}e^{-inx} \, dx - \int_{-\pi}^{\pi} e^{imx}e^{inx} \, dx \right) = \frac{1}{2} \operatorname{Re} (0 - 0) = 0.$$

Now for any m and n, we have

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = \int_{-\pi}^{\pi} \operatorname{Im} \frac{e^{imx}e^{inx} - e^{imx}e^{-inx}}{2} \, dx$$
$$= \frac{1}{2} \left(\operatorname{Im} \int_{-\pi}^{\pi} e^{imx}e^{inx} \, dx - \operatorname{Im} \int_{-\pi}^{\pi} e^{imx}e^{-inx} \, dx \right) = \frac{1}{2}(0 - 0) = 0,$$

since $\operatorname{Im} \int_{-\pi}^{\pi} e^{imx}e^{-inx} \, dx = 0$ whether $m = n$ or $m \neq n$.
Finally, with $m = n \neq 0$, we have

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{imx}e^{-imx} + e^{imx}e^{imx}}{2} \, dx$$
$$= \frac{1}{2} \operatorname{Re} \left(\int_{-\pi}^{\pi} e^{imx}e^{-imx} \, dx + \int_{-\pi}^{\pi} e^{imx}e^{imx} \, dx \right) = \frac{1}{2} \operatorname{Re} (2\pi + 0) = \pi.$$

Similarly,

$$\int_{-\pi}^{\pi} \sin^2 mx \, dx = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{imx}e^{-imx} - e^{imx}e^{imx}}{2} \, dx$$
$$= \frac{1}{2} \operatorname{Re} \left(\int_{-\pi}^{\pi} e^{imx}e^{-imx} \, dx - \int_{-\pi}^{\pi} e^{imx}e^{imx} \, dx \right) = \frac{1}{2} \operatorname{Re} (2\pi - 0) = \pi.$$

Solutions to Exercises 3.3

1. An antiderivative of $f(z) = z^2+z-1$ is simply $F(z) = \frac{1}{3}z^3 + \frac{1}{2}z^2 - z + C$ where C is an arbitrary complex constant. The function F is entire and so we can take $\Omega = \mathbb{C}$.

5. To find an antiderivative of $\frac{1}{(z-1)(z+1)}$, we proceed as we would have done in calculus. Using partial fractions, write

$$\begin{aligned}
\frac{1}{(z-1)(z+1)} &= \frac{A}{z-1} + \frac{B}{z+1} \\
\frac{1}{(z-1)(z+1)} &= \frac{A(z+1)+B(z-1)}{(z-1)(z+1)} \\
1 &= A(z+1)+B(z-1).
\end{aligned}$$

Taking $z = -1$, it follows that $B = -\frac{1}{2}$. Taking $z = 1$, it follows that $A = \frac{1}{2}$. Hence

$$\frac{1}{(z-1)(z+1)} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}.$$

An antiderivative of this function is

$$F(z) = \frac{1}{2}\big(\operatorname{Log}(z-1) - \operatorname{Log}(z+1)\big) + C,$$

where C is an arbitrary complex constant. (You could also use a different branch of the logarithm.) The function $\operatorname{Log}(z-1)$ is analytic in $\mathbb{C} \setminus (-\infty, 1]$, while the function $\operatorname{Log}(z+1)$ is analytic in $\mathbb{C}\setminus(-\infty,-1]$. So the function $F(z) = \frac{1}{2}\big(\operatorname{Log}(z-1) - \operatorname{Log}(z+1)\big) + C$, is analytic in $\mathbb{C}\setminus(-\infty, 1]$ and we may take $\Omega = \mathbb{C}\setminus(-\infty,1]$. In fact, the function $\operatorname{Log}(z-1) - \operatorname{Log}(z+1) = \operatorname{Log}\frac{z-1}{z+1}$ is analytic in a larger region $\mathbb{C}\setminus[-1,1]$. There are at least two possible ways to see this. One way is to note that the linear fractional transformation $w = \frac{z-1}{z+1}$ takes the interval $[-1, 1]$ onto the half-line $(-\infty, 0]$. All other values of z outside the interval $[-1, 1]$ are mapped into $\mathbb{C}\setminus(-\infty, 0]$ and so the composition $\operatorname{Log}\frac{z-1}{z+1}$ is analytic everywhere on $\mathbb{C}\setminus[-1, 1]$. Another way to show that $\operatorname{Log}(z-1) - \operatorname{Log}(z+1)$ is analytic in $\mathbb{C}\setminus[-1,1]$ is to use Theorem 4, Sec. 2.3. Let $g(z) = \operatorname{Log}(z-1) - \operatorname{Log}(z+1)$ and $f(z) = e^z$. The function $g(z)$ is continuous on $\mathbb{C}\setminus[-1,1]$, because the discontinuities of $\operatorname{Log}(z-1)$ and $\operatorname{Log}(z+1)$ cancel on $(-\infty, -1)$. The function $f(z) = e^z$ is obviously entire. The composition $f(g(z))$ is equal to the function $\frac{z-1}{z+1}$, which is analytic except at the points $z = \pm 1$. According to Theorem 4, Sec. 2.3, the function $g(z)$ is analytic in $\mathbb{C}\setminus[-1,1]$.

9. An antiderivative of $z \sinh z^2$ is $\frac{1}{2}\cosh z^2 + C$, as you can verify by differentiation. The antiderivative is valid for all z.

13. Using Exercise 12, we find that an antiderivative of the function $\log_0 z - \log_{\frac{\pi}{2}} z$ is $z\log_0 z - z\log_{\frac{\pi}{2}} z$. This antiderivative is valid except on the branch cuts of the two logarithms, that is, on $\Omega = \mathbb{C}\setminus\big([0,\infty) \cup \{it : 0 \le t < \infty\}\big)$.

17. The integrand is entire and has an antiderivative in a region containing the path. So, by

Theorem 3.3.4,

$$\begin{aligned}
\int_\gamma z^2\,dz &= \frac{1}{3}z^3\Big|_{\gamma(0)}^{\gamma(\frac{\pi}{4})} = \frac{1}{3}\left((e^{i\frac{\pi}{4}}+3e^{2i\frac{\pi}{4}})^3-(e^{i\cdot 0}+3e^{2i\cdot 0})^3\right)\\
&= \frac{1}{3}\left(e^{i\frac{3\pi}{4}}+9e^{i\frac{2\pi}{4}}e^{i\frac{\pi}{2}}+27e^{i\frac{\pi}{4}}e^{2i\frac{\pi}{2}}+27e^{i\frac{3\pi}{2}}-(4)^3\right)\\
&= \frac{1}{3}\left(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}-9-27(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})-27i-(4)^3\right)\\
&= \frac{1}{3}\left((-14\sqrt{2}-73+i(-13\sqrt{2}-27)\right)
\end{aligned}$$

21. Since $\sin z$ is continuous on the complex plane with entire antiderivative $-\cos z$, we have, by Theorem 3.3.4,

$$\int_\gamma \sin z\mathrm{d}z = -\cos z\Big|_2^{2e^{i\frac{\pi}{2}}} = \cos 2-\cos(2i).$$

25. The function $z\,\mathrm{Log}\,z$ has an antiderivative $\frac{1}{2}z^2\big(\mathrm{Log}\,z-\frac{1}{2}\big)$, and this antiderivative is analytic on the region $\mathbb{C}\setminus(-\infty,0]$, which contains the closed path $[z_1,\,z_2,\,z_3,\,z_1]$. Thus, by Theorem 3.3.4,

$$\int_{[z_1,\,z_2,\,z_3,\,z_1]} z\,\mathrm{Log}\,z\,dz = 0.$$

29. (a) Differentiating, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\alpha+1}z^{\alpha+1} = \frac{1}{\alpha+1}(\alpha+1)z^{\alpha+1-1} = z^\alpha.$$

(b) By part (a), an antiderivative of $\frac{1}{\sqrt{z}} = z^{-\frac{1}{2}}$ is $2z^{\frac{1}{2}} = 2\sqrt{z}$, and this is analytic on $\mathbb{C}\setminus(-\infty,0]$. Since γ is contained in this region, by Theorem 3.3.4, we have

$$\int_\gamma \frac{1}{\sqrt{z}}\,dz = 2\sqrt{z}\Big|_{-i}^{i} = 2\sqrt{2}i.$$

33. (a) Parametrize $C_R(z_0)$ by $\gamma(t) = z_0+Re^{it}, 0\le t\le 2\pi$. Then $\gamma'(t) = iRe^{it}$ and we have

$$\int_\gamma \mathrm{Im}\,z\,dz = \int_0^{2\pi}\mathrm{Im}\,(Re^{it}+z_0)iRe^{it}\,dt = \int_0^{2\pi}(R\sin t+\mathrm{Im}\,z_0)iRe^{it}\,dt$$

$$= \int_0^{2\pi} iR^2e^{it}\sin t\,dt + \int_0^{2\pi} i\,\mathrm{Im}\,(z_0)Re^{it}\,dt = \int_0^{2\pi} iR^2\sin t(\cos t+i\sin t)\,dt + \overbrace{\left[\mathrm{Im}\,(z_0)Re^{it}\right]_0^{2\pi}}^{=0}$$

$$= iR^2\int_0^{2\pi}\sin t\cos t\,dt - R^2\int_0^{2\pi}\sin^2 t\,dt = iR^2\int_0^{2\pi}\frac{1}{2}\sin(2t)\,dt - R^2\int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{2}\cos(2t)\right)dt$$

$$= \overbrace{-i\frac{R^2}{4}[\cos(2t)]_0^{2\pi}}^{=0} - \frac{R^2}{2}[t]_0^{2\pi} + \overbrace{\frac{R^2}{4}[\sin(2t)]_0^{2\pi}}^{=0} = -\pi R^2.$$

(b) If $\mathrm{Im}\,z$ had an analytic antiderivative on an open subset Ω of $\mathbb{C}$, there would be some $z_0\in\Omega$ and $R>0$ such that $C_R(z_0)\subset\Omega$. By Theorem 3.3.4, we should have $\int_{C_R(z_0)}\mathrm{Im}\,z\,dz = 0$, but by (a), we have $\int_{C_R(z_0)} \operatorname{Im} z \, dz = -\pi R^2 \neq 0$, a contradiction. Therefore, no such antiderivative can exist.

(c) If $\operatorname{Re} z$ had an analytic antiderivative on an open subset Ω of $\mathbb{C}$, then, since z has an entire antiderivative, we would also have an antiderivative for $\operatorname{Im} z = \frac{z - \operatorname{Re} z}{i}$ on Ω, but by (b), such an antiderivative cannot exist. Thus, $\operatorname{Re} z$ cannot have such an antiderivative.

Solutions to Exercises 3.4

1. Let $\gamma_1, \ldots, \gamma_4$ be the four sides of the square, parametrized as

$$\begin{aligned} &x_1(t) = t, && y_1(t) = 0, \\ &x_2(t) = 1, && y_2(t) = t, \\ &x_3(t) = 1 - t, && y_3(t) = 1, \\ &x_4(t) = 0, && y_4(t) = 1 - t, \\ &&0 \le t \le 1. \end{aligned}$$

Then the left integral from (3.4.1) becomes

$$\int_\gamma P\, dx + Q\, dy$$
$$= \int_{\gamma_1} xy\, dx + y\, dy + \int_{\gamma_2} xy\, dx + y\, dy + \int_{\gamma_3} xy\, dx + y\, dy + \int_{\gamma_4} xy\, dx + y\, dy.$$

Parametrizing each integral, we obtain

$$\begin{aligned} \int_{\gamma_1} xy\, dx + y\, dy &= \int_0^1 0 + 0\, dt = 0, \\ \int_{\gamma_2} xy\, dx + y\, dy &= \int_0^1 0 + t\, dt = \frac{1}{2}, \\ \int_{\gamma_3} xy\, dx + y\, dy &= \int_0^1 (1-t)(1)(-1) + 0\, dt = -\frac{1}{2}, \\ \int_{\gamma_4} xy\, dx + y\, dy &= \int_0^1 0 + (1-t)(-1)\, dt = -\frac{1}{2}. \end{aligned}$$

Thus,

$$\int_\gamma xy\, dx + y\, dy = -\frac{1}{2}.$$

The right integral from (3.4.1) becomes

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_0^1 \int_0^1 -x\, dx\, dy = -\int_0^1 x\, dx \int_0^1 dy = -\frac{1}{2}.$$

These two integrals are equal, verifying Green's Theorem.

5. The area of D is most straightforwardly expressed as

$$\iint_D dx\, dy.$$

For the first integral, we have $P = -y, Q = 0$, so by Green's Theorem, we have

$$\int_\gamma -y\, dx = \iint_D (0 - (-1))\, dx\, dy = \iint_D dx\, dy.$$

For the second, we have $P = 0, Q = x$, which by Green's Theorem gives

$$\int_\gamma x\, dy = \iint_D (1 - 0)\, dx\, dy = \iint_D dx\, dy.$$

Finally, the third integral is equal to the average of the first two, so it is also equal to the area of D.

9. (a) For $t \neq 0$, we have

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t} t^2 \sin\frac{1}{t} = 2t \sin\frac{1}{t} - \cos\frac{1}{t},$$

and this is defined for all $t \neq 0$. At $t = 0$, by the definition of the derivative, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x \sin\frac{1}{x} = 0.$$

Thus, $f'(t)$ is defined for all $t \in \mathbb{R}$.

(b) To show that the given curve is a path, first note that the curve is continuous: the graph of $f(t)$ is clearly continuous for $t \neq 0$, and $\lim_{t\to 0} f(t) = 0 = f(0)$ by the squeeze theorem, so the portion of the curve from $f(t)$ is continuous. Furthermore, since $f(0) = 0$ and $f(\frac{1}{\pi}) = 0$, the curve is continuous at the piece boundaries. The straight line segment is clearly continuous. Then, to show that the curve is piecewise-continuously differentiable, note that we can parametrize the part of the curve from $f(t)$ by $\gamma(t) = t + if(t)$, $0 \leq t \leq \frac{1}{\pi}$. We then have $\gamma'(t) = 1 + f'(t)$, and as $f'(t)$ exists from part (a) and is continuous on $\left(0, \frac{1}{\pi}\right)$, this piece of the curve is continuously differentiable. The straight line segment from $\left(0, \frac{1}{\pi}\right)$ to $(0, 0)$ is clearly continuously differentiable. Thus, the curve is piecewise-continuously differentiable, and, thus, a path. Since $f(0) = 0$ and the straight line ends at $(0, 0)$, the curve is clearly closed. To see that the path intersects itself an infinite number of times, note that for any $n \in \mathbb{N}$, $t_n := \frac{1}{n\pi} \in \left[0, \frac{1}{\pi}\right]$ and $f(t_n) = 0$. Thus, the piece of the curve from $f(t)$ intersects the straight line segment at an infinite number of points $(t_n, 0)$.

Solutions to Exercises 3.5

1. This is both convex and star-shaped.

5. This is neither convex nor star-shaped.

9. The integrand is analytic on $\mathbb{C} \setminus \{-2, 2\}$, and the path of integration and its interior lie inside a star-shaped region on which the integrand is analytic (take, for example, $B_{\frac{3}{2}}(0)$). By Theorem 3.5.4, the integral over the square, traced once, is zero. Since the integral over the given path is ten times this, it is also zero.

Solutions to Exercises 3.6

1. Each path is continuously deformable to a point in Ω and Ω is connected. So the two paths are homotopic.

5. The region Ω is simply connected, so the two paths are homotopic.

9. Given the path γ, consider the homotopy $H(t,s) = \gamma\big((1-s)t\big)$. Then $H(t,0) = \gamma(t)$ and $H(t,1) = \gamma(0)$. Obviously H is continuous, so γ is homotopic to a point.

13. (a) The first figure is convex, the second and third are not.

(b) We have trivially that $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$. Since $\gamma_0(t)$ and $\gamma_1(t)$ are continuous in t, and s and $1-s$ are continuous in s, $H(t,s)$ is continuous in both variables. Furthermore, since $\gamma_0(t_0)$ and $\gamma_1(t_0)$ are in Ω for all $0 \le t_0 \le 1$ and Ω is convex, we have that $H(t_0,s) = (1-s)\gamma_0(t_0) + s\gamma_1(t_0) \in \Omega$ for all $0 \le s \le 1$. Thus, $H(t,s)$ is a continuous mapping from the unit square into Ω and is a homotopy from γ_0 to γ_1.

(c) The same mapping (3.6.17) and the same reasoning as (b) shows this.

(d) Let $\gamma_1(t) = z_1$ in part (c).

(e) Using (3.6.17), we have

$$H(t,s) = (1-s)\big(e^{2\pi it} + e^{4\pi in}\big) + 3se^{2\pi it}.$$

Solutions to Exercises 3.7

1. By Example 3.7.4, we have
(a)

$$\int_C \frac{2i}{z-i}\,dz = 2i\int_C \frac{1}{z-i} = -4\pi.$$

(b)

$$\int_C \frac{2i}{z-i}\,dz = 2i\int_C \frac{1}{z-i} = 0.$$

5. We have

$$\int_C \left[\frac{2i}{z-2} - \frac{3+2i}{z+i}\right] dz = 2i\int_C \frac{1}{z-2} - (3+2i)\int_C \frac{1}{z+i}\,dz.$$

Thus,
(a)

$$\int_C \left[\frac{2i}{z-2} - \frac{3+2i}{z+i}\right] dz = 2i(2\pi i) - 0 = -4\pi.$$

(b)

$$\int_C \left[\frac{2i}{z-2} - \frac{3+2i}{z+i}\right] dz = 2i(2\pi i) - (3+2i)(2\pi i) = -6\pi i.$$

(c)

$$\int_C \left[\frac{2i}{z-2} - \frac{3+2i}{z+i}\right] dz = 0 - 0 = 0.$$

There is one overlooked combination, where 2 is outside C and $-i$ is inside C:

$$\int_C \left[\frac{2i}{z-2} - \frac{3+2i}{z+i}\right] dz = 0 - (3+2i)(2\pi i) = 4\pi - 6\pi i.$$

9. Since 1 lies in the interior of γ,

$$\int_\gamma \frac{dz}{z-1} = 2\pi i.$$

13. Since $\dfrac{e^z}{z+2}$ is analytic on $C_1(0)$ and its interior, then the integral can be computed as

$$\int_{C_1(0)} \frac{e^z}{z+2}\mathrm{d}z = 0.$$

17. The function $f(z) = \frac{e^z}{z+i}$ has a discontinuity at $z = -i$. Hence, it is analytic inside an on the simple path $\gamma(t) = i + e^{it}$, $0 \le t \le 2\pi$ (circle, centered at i with radius 1). By Theorem 3.6.7,

$$\int_\gamma \frac{e^z}{z+i}\,dz = 0.$$

21. The path $[z_1, z_2, z_3, z_1]$ is contained in a region that does not intersect the branch cut of $\operatorname{Log} z$. Hence the function $f(z) = z^2 \operatorname{Log} z$ is analytic inside an on the simple path $[z_1, z_2, z_3, z_1]$, and so by Theorem 5,

$$\int_{[z_1, z_2, z_3, z_1]} z^2 \operatorname{Log} z\,dz = 0.$$

25. The path $C_2(0)$ contains both roots of the polynomial $z^2 - 1$. We will evaluate the integral by using the method of Example 5. We have

$$\frac{z}{z^2-1} = \frac{A}{z-1} + \frac{B}{z+1} \qquad \Rightarrow \qquad z = A(z+1) + B(z-1).$$

Setting $z = 1$, we get $1 = 2A$ or $A = \frac{1}{2}$. Setting $z = -1$, we get $-1 = -2B$ or $B = \frac{1}{2}$. Hence

$$\frac{z}{z^2-1} = \frac{1}{2(z-1)} + \frac{1}{2(z+1)},$$

and so

$$\int_{C_2(0)} \frac{z}{z^2-1}dz = \frac{1}{2}\int_{C_2(0)} \frac{1}{z-1}dz + \frac{1}{2}\int_{C_2(0)} \frac{1}{z+1}dz = \frac{1}{2}2\pi i + \frac{1}{2}2\pi i = 2\pi i,$$

where we have applied the result of Example 4 in evaluating the integrals.

29. Write $\gamma = (\gamma_1, \gamma_2)$, where γ_1 is the circle centered at i and γ_2 is the circle centered at -1. Then

$$\int_\gamma \frac{1}{(z+1)^2(z^2+1)}\,dz = \int_{\gamma_1} \frac{1}{(z+1)^2(z^2+1)}\,dz + \int_{\gamma_2} \frac{1}{(z+1)^2(z^2+1)}\,dz = I_1 + I_2.$$

The partial fraction decomposition of the integrand is

$$\frac{1}{(z+1)^2(z^2+1)} = \frac{1}{2(z+1)} + \frac{1}{2(z+1)^2} - \frac{1}{4(z+i)} - \frac{1}{4(z-i)}.$$

We have

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\gamma_1} \frac{dz}{z+1} + \frac{1}{2}\int_{\gamma_1} \frac{dz}{(z+1)^2} - \frac{1}{4}\int_{\gamma_1} \frac{dz}{z+i} - \frac{1}{4}\int_{\gamma_1} \frac{dz}{z-i} \\ &= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 - \frac{1}{4}\cdot 0 - \frac{1}{4}\cdot 2\pi i = -\frac{\pi}{2}i, \end{aligned}$$

where the first 3 integrals are 0 because the integrands are analytic inside an on γ_1, and the fourth integral follows from Example 4. Similarly,

$$\begin{aligned} I_2 &= \frac{1}{2}\int_{\gamma_2}\frac{dz}{z+1} + \frac{1}{2}\int_{\gamma_2}\frac{dz}{(z+1)^2} - \frac{1}{4}\int_{\gamma_2}\frac{dz}{z+i} - \frac{1}{4}\int_{\gamma_2}\frac{dz}{z-i} \\ &= \frac{1}{2}\cdot 2\pi i + \frac{1}{2}\cdot 0 - \frac{1}{4}\cdot 0 - \frac{1}{4}\cdot 0 = \pi i, \end{aligned}$$

where the first, third, and fourth integrals follow from Example 4, and the second integral follows from Example 4, Sec. 3.2. Thus, the desired integral is equal to

$$I_1 + I_2 = \frac{\pi}{2} i.$$

33. Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$. Then we have

$$\begin{gathered} \frac{1}{2\pi i}\int_C \frac{p(z)}{z-z_0}\,dz \\ = a_n \frac{1}{2\pi i}\int_C \frac{z^n}{z-z_0}\,dz + \cdots + a_1 \frac{1}{2\pi i}\int_C \frac{z}{z-z_0}\,dz + a_0 \frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\,dz \\ = a_n z_0^n + \cdots + a_1 z_0 + a_0 \quad \text{[by Exercise 32]} \\ = p(z_0). \end{gathered}$$

37. The integrand factors as

$$\frac{1}{z^4+1} = \frac{1}{\left(z - e^{i\frac{\pi}{4}}\right)\left(z - e^{i\frac{3\pi}{4}}\right)\left(z - e^{i\frac{5\pi}{4}}\right)\left(z - e^{i\frac{7\pi}{4}}\right)},$$

and all of these roots are distinct and inside $C_2(0)$. By Exercise 36, we have

$$\int_{C_2(0)} \frac{1}{z^4+1} = \int_{C_2(0)} \frac{1}{\left(z - e^{i\frac{\pi}{4}}\right)\left(z - e^{i\frac{3\pi}{4}}\right)\left(z - e^{i\frac{5\pi}{4}}\right)\left(z - e^{i\frac{7\pi}{4}}\right)} = 0.$$

Solutions to Exercises 3.8

1. Apply Cauchy's formula with $f(z) = \cos z$ at $z = 0$. Then

$$\int_{C_1(0)} \frac{\cos z}{z}\, dz = \int_{C_1(0)} \frac{\cos z}{z-0}\, dz = 2\pi i f(0) = 2\pi i.$$

5. Apply Cauchy's formula with $f(z) = -\operatorname{Log} z$ at $z = i$. Then

$$\int_{C_{\frac{1}{2}}(i)} \frac{\operatorname{Log} z}{-z+i}\, dz = \int_{C_{\frac{1}{2}}(i)} \frac{-\operatorname{Log} z}{z-i}\, dz = -2\pi i \operatorname{Log} i = -2\pi i \left(\ln 1 + i\frac{\pi}{2}\right) = \pi^2.$$

9. We apply the generalized Cauchy formula with $f(z) = \sin z$ at $z = \pi$ with $n = 2$. Then

$$\int_\gamma \frac{\sin z}{(z-\pi)^3}\, dz = \frac{2\pi i}{2!} f^{(2)}(\pi) = \pi i(-\sin\pi) = 0.$$

13. Follow the solution in Example 2. Draw small nointersecting *negatively* oriented circles inside γ, γ_1 centered at 0 and γ_2 centered at i. Then

$$\int_\gamma \frac{z+\cos(\pi z)}{z(z^2+1)}\, dz = \int_{\gamma_1} \frac{z+\cos(\pi z)}{z(z^2+1)}\, dz + \int_{\gamma_2} \frac{z+\cos(\pi z)}{z(z^2+1)}\, dz = I_1 + I_2.$$

Apply Cauchy's formula with $f(z) = \frac{z+\cos(\pi z)}{z^2+1}$ at $z = 0$. Then (recall γ_1 is negatively oriented)

$$I_1 = \int_{\gamma_1} \frac{z+\cos(\pi z)}{z(z^2+1)}\, dz = -2\pi i f(0) = -2\pi i \frac{0+\cos 0}{0^2+1} = -2\pi i.$$

Apply Cauchy's formula with $f(z) = \frac{z+\cos(\pi z)}{z(z+i)}$ at $z = i$. Then

$$\begin{aligned} I_2 &= \int_{\gamma_2} \frac{z+\cos(\pi z)}{z(z^2+1)}\, dz = \int_{\gamma_2} \frac{z+\cos(\pi z)}{z(z+i)(z-i)}\, dz \\ &= -2\pi i f(i) = -2\pi i \frac{i+\cos\pi i}{i(2i)} = +\pi i(i+\cosh\pi). \end{aligned}$$

So $I_1 + I_2 = -\pi - i\pi(2 - \cosh\pi)$.

17. Factor the denominator as $z^3 - 3z + 2 = (z+2)(z-1)^2$. Apply the generalized Cauchy formula (6), with $f(z) = \frac{1}{z+2}$ at $z = 1$, with $n = 1$. Then

$$\int_{C_{\frac{3}{2}}(0)} \frac{dz}{(z+2)(z-1)^2} = 2\pi i f'(1) = 2\pi i \frac{-1}{3^2} = -\frac{2\pi i}{9}.$$

21. Let $\zeta = e^{it}, 0 \le t \le 2\pi$. Then $d\zeta = ie^{it}\, dt$, and we can reexpress the integral as

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it}-z}\, dt = \frac{1}{2\pi} \int_{C_1(0)} \frac{1}{\zeta - z} \frac{d\zeta}{i}.$$

Since $|z| < 1$, z is within $C_1(0)$, by Cauchy's Theorem, we have

$$F(z) = \frac{1}{2\pi i} \int_{C_1(0)} \frac{1}{\zeta - z}\, d\zeta = 1.$$

25. Define $g(z) = \int_0^1 \cos(zt)\, dt$ and let $\zeta = t$ for $0 \le t \le 1$. Then $g(z)$ becomes $g(z) = \int_{[0,1]} \cos(z\zeta)\, d\zeta$. Let $\phi(z, \zeta) = \cos(z\zeta)$. Then $\phi(z, \zeta)$ is continuous in $\zeta \in [0, 1]$ and analytic in $z \in \mathbb{C}$. Furthermore, the derivative $\frac{\mathrm{d}\phi}{\mathrm{d}z} = \zeta \cos(z\zeta)$ is continuous in $\zeta \in [0, 1]$. By Theorem 3.8.5, $g(z)$ is analytic in $\mathbb{C}$, i.e., entire.

If $z = 0$, $g(z) = \int_0^1 dt = 1$. For $z \neq 0$, we have

$$g(z) = \int_0^1 \cos zt\, dt = \frac{1}{z} \sin zt\Big|_{t=0}^{1} = \frac{\sin z}{z}.$$

Thus

$$g(z) = \begin{cases} 1 & \text{if } z = 0, \\ \frac{\sin z}{z} & \text{if } z \neq 0, \end{cases}$$

is an entire function.

29. For z inside C, by Cauchy's formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

But $f(\zeta) = g(\zeta)$ for ζ on C. So, for all z inside C,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta - z} d\zeta = g(z),$$

by Cauchy's formula applied to g.

33. Since f is analytic at $z = z_0$, then there exists an open set in which f is analytic. Choose $R > 0$ such that $C_R(z_0)$ and its interior are contained in D. Now consider

$$\frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} \mathrm{d}\zeta,$$

for $|z - z_0| < R$. Then

(i) For $z = z_0$. Since $f(z)$ is analytic at $z = z_0$, then $f(\zeta)$ is analytic inside and on $C_R(z_0)$ with the existence of $f'(z_0)$.

$$\frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2} \mathrm{d}\zeta = f'(\zeta)|_{\zeta = z_0} = f'(z_0).$$

(ii) For $z \neq z_0$. Since $\frac{f(\zeta)}{\zeta - z}$ is analytic inside and on $C_\epsilon(z_0)$ and $\frac{f(\zeta)}{\zeta - z_0}$ is analytic inside and on $C_\epsilon(z)$ for some sufficiently small $\epsilon > 0$, then by Cauchy's Integral Formula:

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)}\mathrm{d}\zeta &= \frac{1}{2\pi i}\int_{C_\epsilon(z_0)} \frac{\frac{f(\zeta)}{\zeta - z}}{\zeta - z_0}\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{\frac{f(\zeta)}{\zeta - z_0}}{\zeta - z}\mathrm{d}\zeta \\ &= \left[\frac{f(\zeta)}{\zeta - z}\right]\Big|_{\zeta = z_0} + \left[\frac{f(\zeta)}{\zeta - z_0}\right]\Big|_{\zeta = z} \\ &= \frac{f(z) - f(z_0)}{z - z_0}.\end{aligned}$$

Combining the two statements above, we can obtain:

$$\phi(z) = \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)}\mathrm{d}\zeta = \begin{cases} f'(z_0) & \text{if } z = z_0 \\ \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0. \end{cases}$$

Since when $z \neq z_0$, we have $\phi(z) = \frac{f(z)-f(z_0)}{z-z_0}$ is clearly analytic at z_0. For $z = z_0$, first denote

$$F(z, \zeta) := \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)},$$

thus $F(z, \zeta)$ is analytic with respect to z. Thus $\frac{\mathrm{d}}{\mathrm{d}z}F(z, \zeta)$ exists for all $|z - z_0| < R$. Given any fixed $r > 0$ with $0 < |z - z_0| < r < R$. Since F is analytic inside and on $C_r(z_0)$ which implies that F is continuous on $C_r(z_0)$ and its interior, then there exists $M > 0$ such that M is the maximum of $|F(z, \zeta)|$ for all z on $C_r(z_0)$ and its interior and $\zeta \in C_R(z_0)$, thus for $0 < |z - z_0| < \frac{r}{2}$:

$$\Big|\frac{F(z, \zeta) - F(z_0, \zeta)}{z - z_0} - \frac{\mathrm{d}}{\mathrm{d}z}F(z_0, \zeta)\Big| \leq \frac{2M|z - z_0|}{r^2}.$$

Integrating the expression inside the absolute value on the left, we have

$$\Big|\frac{\phi(z) - \phi(z_0)}{z - z_0} - \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{\mathrm{d}}{\mathrm{d}z}F(z_0, \zeta)\mathrm{d}\zeta\Big| \leq \frac{2\pi R}{2\pi}\frac{2M|z - z_0|}{r^2} = \frac{2MR|z - z_0|}{r^2} \to 0,$$

as $|z - z_0| \to 0$. Thus $\lim\limits_{z \to z_0} \frac{\phi(z) - \phi(z_0)}{z - z_0} = \frac{1}{2\pi i}\int_{C_R(0)} \frac{\mathrm{d}}{\mathrm{d}z}\frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)}\mathrm{d}\zeta$ exists, which implies that $\phi(z)$ is analytic at $z = 0$. Since $r < R$ is arbitrarily chosen, we have $\phi(z)$ is analytic for $|z - z_0| < R$. Therefore, ϕ is analytic at z_0.

Solutions to Exercises 3.9

1. We have $|f(z)| = |z|$. Obviously, if z belongs to the unit disk, $|z| \le 1$, then the largest value of $|f(z)|$ is 1 and its smallest value is 0. Hence $|f(z)|$ attains its maximum value at <u>all</u> points of the boundary, and it attains its minimum value at the point $z = 0$, inside the region. The fact that the minimum is attained inside the region does not contradict Corollary 3, because $f(z)$ vanishes at $z = 0$ inside of Ω.

5. The function $f(z) = \frac{z}{z^2+2}$ is continuous for all z such that $2 \le |z| \le 3$ and does not vanish inside this annular region. Thus according to Corollary 3, $|f|$ attains its maximum and minimum at the boundary; that it at points where $|z| = 2$ or $|z| = 3$. Using the triangle inequality, we have

$$|z^2 + 2| \le |z|^2 + 2 \quad \Rightarrow \quad \frac{1}{|z|^2 + 2} \le \frac{1}{|z^2 + 2|};$$
$$|z^2 + 2| \ge \big||z|^2 - 2\big| \quad \Rightarrow \quad \frac{1}{|z^2 + 2|} \le \frac{1}{\big||z|^2 - 2\big|}.$$

On the part of the boundary $|z| = 2$, we have

$$\frac{1}{3} = \frac{2}{|2|^2 + 2} \le |f(z)| = \left|\frac{z}{z^2 + 2}\right| \le \frac{2}{|2|^2 - 2|} = 1.$$

On the part of the boundary $|z| = 3$, we have

$$\frac{3}{11} = \frac{3}{|3|^2 + 2} \le |f(z)| = \left|\frac{z}{z^2 + 2}\right| \le \frac{3}{|3|^2 - 2|} = \frac{3}{7}.$$

Thus the smallest value of $|f(z)|$ is $\frac{3}{11}$. It is attained at a point z with $|z| = 3$. For this value of z, we must have $|z^2 + 2| = 11$. The only possibilities are $z = \pm 3$.

The largest value of $|f(z)|$ is 1. It is attained at a point z with $|z| = 2$. For this value of z, we must have $|z^2 + 2| = 2$ or $z^2 = -4$. The only possibilities are $z = \pm 2i$.

9. We have

$$\begin{aligned} |f(z)| &= |\ln |z| + i \operatorname{Arg} z|; \\ |f(z)|^2 &= (\ln |z|)^2 + (\operatorname{Arg} z)^2. \end{aligned}$$

The largest value (respectively, minimum value) of $|f(z)|$ is attained when $|f(z)|^2$ attains its largest value (respectively, minimum value). The largest value of $|f(z)|^2 = (\ln |z|)^2 + (\operatorname{Arg} z)^2$ is clearly attained when $|z| = 2$ and $\operatorname{Arg} z = \frac{\pi}{4}$. So $|f(z)|$ attains its maximum value $\sqrt{(\ln 2)^2 + (\frac{\pi}{4})^2}$ when $z = 2e^{i\frac{\pi}{4}}$.

The smallest value of $|f(z)|^2 = (\ln |z|)^2 + (\operatorname{Arg} z)^2$ is clearly attained when $|z| = 1$ and $\operatorname{Arg} z = 0$. So $|f(z)|$ attains its smallest value 0 when $z = 1$.

13. (a) To verify the identity

$$z^n - w^n = (z - w)(z^{n-1} + z^{n-2}w + z^{n-3}w^2 + \cdots + z\,w^{n-2} + w^{n-1}),$$

expand the right side. All terms cance except for $z^n - w^n$.

(b) If $p(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0$ is a polynomial of degree $n \geq 2$, and if $p(z_0) = 0$, then $p_n z_0^n + p_{n-1} z_0^{n-1} + \cdots + p_1 z_0 + p_0 = 0$. From (a),

$$\begin{aligned} p(z) = p(z) - p(z_0) &= p_n(z^n - z_0^n) + p_{n-1}(z^{n-1} - z_0^{n-1}) + \cdots + p_1(z - z_0) \\ &= (z - z_0) q(z), \end{aligned}$$

where $q(z) = (z^{n-1} + z^{n-2} z_0 + z^{n-3} z_0^2 + \cdots + z\, z_0^{n-2} + z_0^{n-1})$ is a polynomial of degree $n-1$ in z.

17. Suppose that f is entire and that it omits an open nonempty set, say, there is an open disk $B_R(w_0)$ with $R > 0$ in the w-plane such that $f(z)$ is not in $B_R(w_0)$ for all z. Let $g(z) = \frac{1}{f(z)-w_0}$. Then $g(z)$ is also entire, because $f(z) \neq w_0$ for all z. In fact, since $f(z)$ is not in $B_R(w_0)$, its distance to w_0 is always greater than R. That is, $|f(z) - w_0| \geq R$. But this implies that $|g(z)| \leq \frac{1}{R}$, which in turn implies that g is constant, by Liouville's theorem. Since $g(z) \neq 0$, this constant is obviously not 0. So $C = \frac{1}{f(z)-w_0}$, hence $f(z) = \frac{1}{C} + w_0$ is constant.

21. Suppose that $f(z) = f(x + i\,y)$ is periodic in x and y, and let $T_1 > 0$ and $T_2 > 0$ be such that $f((x + T_1) + i\,(y + T_2)) = f(x + i\,y)$ for all $z = x + i\,y$. Because f is periodic in both x and y, its values repeat on every $T_1 \times T_2$-rectangle. This means that, if we take the rectangle $R = [0, T_1] \times [0, T_2]$ and consider the values of f on this rectangle, then these are all the values taken by $f(z)$, for z in $\mathbb{C}$. The reason is that the complex plane can be tiled by translates of R in the x and y direction, where in the x drection we translate by T_1 units at a time, and in the y direction we translate by T_2 units at a time. Now since f is continuous, it is bounded on R; that is $|f(z)| \leq M$ for some constant M and all z in R. But since f takes on all its values in R, we concude that $|f(z)| \leq M$ for all z in $\mathbb{C}$, and thus f must be constant by Liouville's theorem.

If f is constant in x or y alone, f need not be constant. As an example, take $f(z) = \sin z$ then f is 2π-periodic in the x variable. By considering $f(iz) = \sin(iz)$, we obtain a function that is periodic in y alone. Both functions are entire and clearly not constant.

25. Let $g(z) = e^{if(z)}$, for all $|z| \leq 1$. Since f is analytic on $|z| < 1$ and continuous on $|z| \leq 1$, then g is also analytic on $|z| < 1$ and continuous on $|z| \leq 1$. Since f is real-valued for all $|z| = 1$, we have that $|g(z)| = 1$ for all $|z| = 1$. Since g is nonvanishing in $|Z| < 1$, by Exercise 23, we have $g(z)$ is constant on $|z| \leq 1$. Thus

$$g(z) = e^{if(z)} = A, \quad \text{where } |A| = 1.$$

This implies that $f(z) = c + 2k(z)\pi$ for some $c \in \mathbb{R}$ and $k : \mathbb{C} \to \mathbb{Z}$. Since $f(z)$ is continuous on $|z| \leq 1$, we must have that $k(z) = k_0$ and $f(z)$ is constant on $|z| \leq 1$.

Solutions to Exercises 4.1

1. The sequence of functions $f_n(x) = \dfrac{\sin nx}{n}$ converges uniformly on the interval $0 \le x \le \pi$, with $\lim_{n\to\infty} f_n(x) = 0$ for all $0 \le x \le \pi$. To see this, let $M_n = \max|0 - f_n(x)| = \max|f_n(x)|$, where the maximum is taken over all x in $[0, \pi]$. Then $M_n =\le \frac{1}{n}$. Since $M_n \to 0$, as $n \to \infty$, it follows that f_n converges uniformly to $f = 0$ on $[0, \pi]$. In fact, we have uniform convergence on the entire real line.

5. (a) and (b) First, let us determine the pointwise limit of the sequence of functions $f_n(x) = \dfrac{nx}{n^2x^2 - x + 1}$. For x in the interval $0 \le x \le 1$, we have

$$f_n(x) = \frac{nx}{n^2x^2 - x + 1} = \frac{nx}{n^2(x^2 - \frac{x}{n^2} + \frac{1}{n^2}} = \frac{x}{n(x^2 - \frac{x}{n^2} + \frac{1}{n^2}} \to 0, \text{ as } n \to \infty.$$

Does the sequence converge to 0 uniformly for all x in $[0, 1]$? To answer this question we estimate the maximum possible difference between 0 and $f_n(x)$, as x varies in $[0, 1]$. For this purpose, we compute $M_n = \max|f_n(x)|$ for x in $[0, 1]$. We have

$$f_n'(x) = \frac{n - n^3\,x^2}{(1 - x + n^2\,x^2)^2}; \qquad f_n'(x) = 0 \to n - n^3\,x^2 = 0 \to x = \frac{1}{n};$$
$$f_n(\frac{1}{n}) = \frac{1}{2 - \frac{1}{n}} > \frac{1}{2} \quad \to \quad M_n > \frac{1}{2}.$$

Since M_n does not converge to 0, we conclude that the sequence does not converge uniformly to 0 on $[0, 1]$.
(c) The sequence does converge uniformly on any interval of the form $[a, b]$, where $0 < a < b \le 1$. To see this, pick n so that $0 < \frac{1}{n} < a$. Then, $f_n(x) < 0$ for all $a < x$ (check the sign of $f_n'(x)$ if $\frac{1}{n} < x$. Hence $f_n(x)$ is decreasing on the interval $[a, b]$. So, if $M_n = \max|f_n(x)|$ for x in $[a, b]$, then $0 \le M_n \le |f_n(a)|$. But $f_n(a) \to 0$, by part (a), so thus $M_n \to 0$, and so $f_n(x)$ converges uniformly on $[a, b]$.

9. (a) We have $f_n(0) = \dfrac{1}{2n^2} \to 0$, as $n \to \infty$. For any $0 < |z| \le 1$,

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{nz + 1}{z + 2n^2} = \lim_{n\to\infty} \frac{\frac{z}{n} + \frac{1}{n^2}}{\frac{z}{n^2} + 2} = 0.$$

Therefore, for all $z \in \mathbb{C}$ with $|z| \le 1$, the sequence $\{f_n(z)\}$ converges pointwisely to $f(z) = 0$.
(b) Let $n > 1$, we have for all z with $|z| \le 1$:

$$|f_n(z) - f(z)| = \left|\frac{nz + 1}{z + 2n^2}\right| \le \frac{n|z| + 1}{2n^2 - |z|} \le \frac{n + 1}{2(n^2 - 1)} = \frac{1}{2(n - 1)} \to 0,$$

as $n \to \infty$. Therefore, for all $z \in \mathbb{C}$ with $|z| \le 1$, the sequence $\{f_n(z)\}$ converges uniformly to $f(z) = 0$.

13. If $|z| \le 1$, then

$$\left|\frac{z^n}{n(n + 1)}\right| \le \frac{1}{n(n + 1)}.$$

Apply the Weierstrass M-test with $M_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Since $\sum M_n$ is a convergent telescoping series, it follows that the series $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$, converges uniformly for all $|z| \leq 1$.

17. If $|z| \leq 2$, then

$$\left| \left(\frac{z+2}{5} \right)^n \right| \leq \left(\frac{4}{5} \right)^n.$$

Apply the Weierstrass M-test with $M_n = \left(\frac{4}{5}\right)^n$. Since $\sum M_n$ is convergent (a geometric series with $r < 1$), it follows that the series $\sum_{n=0}^{\infty} \left(\frac{z+2}{5} \right)^n$ converges uniformly for all $|z| \leq 2$.

21. If $2.01 \leq |z - 2| \leq 2.9$, then

$$\left| \frac{(z-2)^n}{3^n} \right| \leq \left(\frac{2.9}{3} \right)^n = A_n,$$

and

$$\left| \frac{2^n}{(z-2)^n} \right| \leq \left(\frac{2}{2.01} \right)^n = B_n.$$

Apply the Weierstrass M-test with $M_n = (A_n + B_n)$. Since $\sum M_n$ is convergent (two geometric series with ratios < 1), it follows that the series $\sum_{n=0}^{\infty} \left\{ \frac{(z-2)^n}{3^n} + \frac{2^n}{(z-2)^n} \right\}$ converges uniformly in the annular region $2.01 \leq |z - 2| \leq 2.9$.

25. (a) For $\left|z - \frac{1}{2}\right| < \frac{1}{6}$, we have

$$|z| = \left| \left(z - \frac{1}{2} \right) + \frac{1}{2} \right| \leq \left| z - \frac{1}{2} \right| + \left| \frac{1}{2} \right| < \frac{2}{3}.$$

Since $\sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n$ is convergent, we conclude from the Weierstrass M-test that $\sum_{n=0}^{\infty} z^n$ converges uniformly on $\left|z - \frac{1}{2}\right| < \frac{1}{6}$.

(b) We claim that $\sum_{n=0}^{\infty} z^n$ does not converge uniformly on $|z - \frac{1}{2}| < \frac{1}{2}$: We have that, for $|z - \frac{1}{2}| < \frac{1}{2}$,

$$|z| = \left| \left(z - \frac{1}{2} \right) + \frac{1}{2} \right| \leq \left| z - \frac{1}{2} \right| + \left| \frac{1}{2} \right| < 1,$$

and we know that the series $\sum_{n=0}^{\infty} z^n$ converges pointwise to $s(z) = \frac{1}{1-z}$ in $|z| < 1$. Its nth partial sum is $s_n(z) = \frac{1 - z^{n+1}}{1-z}$. Take $z = x$ to be a real number with $|x| < 1$. Then

$$|s(x) - s_n(z)| = \left| \frac{x^{n+1}}{1-x} \right| = \frac{x^{n+1}}{1-x} \to \infty \quad \text{as} \quad x \to 1.$$

Therefore, the maximum difference M_n between each partial sum and the series sum is unbounded on $|z| < 1$, so the sequence of partial sums, and, hence, the series, does not converge uniformly on $|z| < 1$.

29. (a) Let $\delta > 1$ be a positive real number. To show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{(principal branch of } n^z\text{)}$$

converges uniformly on the half-plane $H_\delta = \{z : \operatorname{Re} z \geq \delta > 1\}$, we will apply the Weierstrass M-test. For all $z \in H_\delta$, we have

$$|n^z| = \left|e^{(x+iy)\ln n}\right| = e^{x \ln n} = n^x > n^\delta.$$

So

$$\left|\frac{1}{n^z}\right| \leq \frac{1}{n^\delta} = M_n.$$

Since $\sum M_n = \sum \frac{1}{n^\delta}$ is a convergent series (because $\delta > 1$), it follows from the Weierstrass M-test that that $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly in H_δ.

(b) Each term of the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is analytic in $H = \{z : \operatorname{Re} z > 1\}$ (in fact, each term is entire). To conclude that the series is analytic in H, it is enough by Corollary 2 to show that the series converges uniformly on any closed disk contained in H. If S is a closed disk contained in H, S is clearly disjoint from the imaginary axis. Let H_δ ($\delta > 0$) be a half-plane containing S. By part (a), the series converges uniformly on H_δ, consequently, the series converges uniformly on S. By Corollary 2, the series is analytic in H. (Note the subtilty in the proof. We did not show that the series converges uniformly on H. In fact, the series does not converge uniformly in H.)

(c) To compute $\zeta'(z)$, according to Corollary 2, we can differentiate the series term-by-term. Write

$$\frac{1}{n^z} = \frac{1}{e^{z \ln n}} = e^{-z \ln n}.$$

Using properties of the exponential function, we have

$$\frac{d}{dz}\frac{1}{n^z} = \frac{d}{dz}e^{-z \ln n} = -\ln n e^{-z \ln n} = -\ln n \frac{1}{n^z}.$$

So, for all $z \in H$,

$$\zeta'(z) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^z}.$$

33. To show that f_n $\{f_n\}$ converges uniformly on Ω, it is enough to show that

$$\max_{z \in \Omega} |f_n(z) - f_m(z)|$$

can be made arbitrarily small by choosing m and n large. In other words, given $\varepsilon > 0$, we must show that there is a positive integer N such that if $m, n \geq N$, then

$$\max_{z \in \Omega} |f_n(z) - f_m(z)| < \varepsilon.$$

This will show that the sequence $\{f_n\}$ is uniformly Cauchy, and hence it is uniformly convergent by Exercise 30.

Since each f_n is analytic inside C and continuous on C, it follows that $f_n - f_m$ is also analytic inside C and continuous on C. Since C is a simple closed path, the region interior to C is a bounded region. By the maximum principle, Corollary 2, Sec. 3.7, the maximum value of $|f_n - f_m|$ occurs on C. But on C the sequence $\{f_n\}$ is a Cauchy sequence, so there is a positive integer N such that if m, $n \geq N$, then

$$\max_{z\in C} |f_n(z) - f_m(z)| < \varepsilon.$$

Hence

$$\max_{z\in\Omega} |f_n(z) - f_m(z)| \leq \max_{z\in C} |f_n(z) - f_m(z)| < \varepsilon,$$

which is what we want to prove.

The key idea in this exercise is that the maximum value of an analytic function occurs on the boundary. So the uniform convergence of a sequence inside a bounded region can be deduced from the uniform convergence of the sequence on the boundary of the region.

Solutions to Exercises 4.2

1. By ratio test, for $z \neq 0$:

$$\rho = \lim_{n\to\infty} \left| (-1)^{n+1} \frac{z^{n+1}}{2n+3} (-1)^n \frac{2n+1}{z^n} \right| = |z| \lim_{n\to\infty} \frac{2n+1}{2n+3} = |z|.$$

Therefore, the series converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. The radius of convergence is 1; the disk of convergence is $|z| < 1$; and the circle of convergence is $|z| = 1$.

5. By ratio test, for $z \neq \frac{1}{2i}$:

$$\rho = \lim_{n\to\infty} \left| \frac{(4iz-2)^{n+1}}{2^{n+1}} \frac{2^n}{(4iz-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(4iz-2)}{2} \right| = |2iz - 1|.$$

Then the series converges absolutely when $|2iz - 1| < 1$, thus

$$\left| 2i\left(z - \frac{1}{2i}\right) \right| \leq 1 \implies \left| z - \frac{(-i)}{2} \right| < \frac{1}{2}.$$

The series diverges when $\left| z - \frac{(-i)}{2} \right| > \frac{1}{2}$. The radius of convergence is $\frac{1}{2}$; the disk of convergence is $\left| z - \frac{(-i)}{2} \right| < \frac{1}{2}$; and the circle of convergence is $\left| z - \frac{(-i)}{2} \right| = \frac{1}{2}$.

9. We compute the radius of convergence by using the Cauchy-Hadamard formula

$$\frac{1}{R} = \limsup \sqrt[n]{\left| \left(1 - e^{in\frac{\pi}{4}}\right)^n \right|} = \limsup \left| 1 - e^{in\frac{\pi}{4}} \right| = 2.$$

To understand why the lim sup is equal to 2, recall that the lim sup is the limit of the sup of the tail of the sequence $\left\{ \left| 1 - e^{in\frac{\pi}{4}} \right| \right\}_{n=N}^{\infty}$, as N tends to ∞. The terms $e^{in\frac{\pi}{4}}$ take values from the set $\left\{ \pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}, \pm i, \pm 1 \right\}$. So the largest value of $\left| 1 - e^{in\frac{\pi}{4}} \right|$ is 2, and this value repeats infinitely often, which explains the value of the lim sup. Thus, $R = \frac{1}{2}$.

13. We have known the geometric series below:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1,$$

then by differentiating term-by-term:

$$\sum_{n=0}^{\infty} nz^{n-1} = \sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}, \quad |z| < 1.$$

It follows that $\displaystyle\sum_{n=1}^{\infty} 2nz^{n-1} = \frac{2}{(1-z)^2}$, where $|z| < 1$, i.e., the radius of convergence is 1.

17. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(3z-i)^n}{3^n} &= \sum_{n=0}^{\infty} \frac{\left(3\left(z-\frac{i}{3}\right)\right)^n}{3^n} = \sum_{n=0}^{\infty}\left(z-\frac{i}{3}\right)^n \\
&= \sum_{n=0}^{\infty} w^n \quad \left(w = z - \frac{i}{3}\right) \\
&= \frac{1}{1-w} = \frac{1}{1-\left(z-\frac{i}{3}\right)} \qquad \left|z-\frac{i}{3}\right| < 1 \\
&= \frac{3}{3+i-3z},
\end{aligned}$$

which is valid for $\left|z-\frac{i}{3}\right| < 1$.

21. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ has a radius of convergence $R_2 > 0$, then both series converge absolutely for $|z-z_0| < \min\{R_1, R_2\}$. Then, by Theorem 1.5.28, we have

$$\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-z_0)^n\right) = \sum_{n=0}^{\infty} c_n(z-z_0)^n, \quad |z-z_0| < \min\{R_1, R_2\},$$

where

$$\begin{aligned}
c_n &= \sum_{k=0}^{n} a_k(z-z_0)^k b_{n-k}(z-z_0)^{n-k} = \sum_{k=0}^{n} a_k b_{n-k}(z-z_0)^n = (z-z_0)^n \sum_{k=0}^{n} a_k b_{n-k} \\
&= (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1}b_1 + a_n b_0)(z-z_0)^n,
\end{aligned}$$

and this series is absolutely convergent on this region. It is possible for this Cauchy product to have a radius of convergence R larger than $\min\{R_1, R_2\}$, but it is guaranteed to be at least this large.

25. (a) In the formula, take $z_1 = z_2 = \frac{1}{2}$, then

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\Gamma(1)\int_0^{\frac{\pi}{2}} \cos^{1-1}\theta \sin^{1-1}\theta d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = 2\frac{\pi}{2} = \pi,$$

so $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
(b) In (9), let $u^2 = t$, $2u\,du = dt$, then

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt = \int_0^{\infty} u^{2(z-1)}e^{-u^2}(2u)du = 2\int_0^{\infty} u^{2z-1}e^{-u^2}du.$$

(c) Using (b)

$$\begin{aligned}
\Gamma(z_1)\Gamma(z_2) &= 2\int_0^{\infty} u^{2z_1-1}e^{-u^2}du 2\int_0^{\infty} v^{2z_2-1}e^{-v^2}dv \\
&= 4\int_0^{\infty}\int_0^{\infty} e^{-(u^2+v^2)}u^{2z_1-1}v^{2z_2-1}du\,dv.
\end{aligned}$$

(d) Switching to polar coordinates: $u = r\cos\theta$, $v = r\sin\theta$, $u^2 + v^2 = r^2$, $dudv = rdrd\theta$; for (u, v) varying in the first quadrant ($0 \le u < \infty$ and $0 \le v < \infty$), we have $0 \le \theta \le \frac{\pi}{2}$, and $0 \le r < \infty$, and the double integral in (c) becomes

$$\begin{aligned}
\Gamma(z_1)\Gamma(z_2) &= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2}(r\cos\theta)^{2z_1-1}(r\sin\theta)^{2z_2-1} r\,dr\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2z_1-1}(\sin\theta)^{2z_2-1} d\theta\, \overbrace{2\int_0^\infty r^{2(z_1+z_2)-1} e^{-r^2} dr}^{=\Gamma(z_1+z_2)} \\
&\qquad \text{(use (b) with } z_1 + z_2 \text{ in place of } z\text{)} \\
&= 2\Gamma(z_1+z_2)\int_0^{\frac{\pi}{2}} (\cos\theta)^{2z_1-1}(\sin\theta)^{2z_2-1} d\theta,
\end{aligned}$$

implying (d).

Solutions to Exercises 4.3

1. According to Theorem 1, the Taylor series around z_0 converges in the largest disk, centered at $z_0 = 0$, in which the function is analytic. Clearly, e^{z-1} is entire, so radius of convergence is $R = \infty$.

5. Since the function $f(z) := \dfrac{z+1}{z-i}$ is analytic for all $z \neq i$, and the Taylor series around $z_0 = 2+i$ converges in the disk centered at z_0 in which the function is analytic, thus the largest disk around z_0 on which f is analytic has radius:

$$\rho = |z_0 - i| = |2 + i - i| = 2.$$

Therefore, the radius of convergence of $f(z)$ at z_0 is $\rho = 2$.

9. We have that

$$\frac{2i}{3-iz} = \frac{2i}{3+i-i(z-(-1))} = \frac{2i}{3+i}\frac{1}{1-\frac{i(z-(-1))}{3+i}},$$

so the Taylor series expansion around $z_0 = -1$ can be written

$$\frac{2i}{3-iz} = \frac{1+3i}{5}\sum_{n=0}^{\infty}\left(\frac{1+3i}{10}\right)^n (z-(-1))^n,$$

with $\left|\frac{1+3i}{10}(z-(-1))\right| < 1$ if and only if $|z-(-1)| < \sqrt{10}$. Therefore, the radius of convergence is $\sqrt{10}$.

13. Arguing as we did in Exercises 1-9, we find that the Taylor series of $f(z) = \dfrac{z}{1-z}$ around $z_0 = 0$ has radius of convergence equal to the distance from $z_0 = 0$ to the nearest point where f fails to be analytic. Thus $R = 1$. (This will also come out of the computation of the Taylor series.) Now, for $|z| < 1$, the geometric series tells us that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

Multiplying both sides by z, we get, for $|z| < 1$,

$$\frac{z}{1-z} = \sum_{n=0}^{\infty} z^{n+1}.$$

17. Because the function is entire, the Taylor series will have an infinite radius of convergence. Note that the series expansion around 0 is easy to obtain:

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \Rightarrow ze^z = z\sum_{n=0}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{n+1}}{n!}.$$

But how do we get the series expansion around $z_0 = 1$? In the previous expansion, replacing z by $z-1$, we get

$$(z-1)e^{z-1} = \sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{n!}.$$

The expansion on the right is a Taylor series centered at $z_0 = 1$, but the function on the left is not quite the function that we want. Let $f(z) = ze^z$. We have

$$(z-1)e^{z-1} = e^{-1}ze^z - e^{z-1} = e^{-1}f(z) - e^{z-1}.$$

So $f(z) = e\big[(z-1)e^{z-1} + e^{z-1}\big]$. Using the expansion of $(z-1)e^{z-1}$ and the expansion of $e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$, we find

$$\begin{aligned}
f(z) &= e\big[\sum_{n=0}^{\infty} \frac{(z-1)^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}\big] = e\big[\sum_{n=1}^{\infty} \frac{(z-1)^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}\big] \\
&= e\big[\sum_{n=1}^{\infty} \frac{(z-1)^n}{(n-1)!} + +1 + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}\big] \\
&= e\big[1 + \sum_{n=1}^{\infty} (z-1)^n \left(\frac{1}{(n-1)!} + \frac{1}{n!}\right)\big] = e\big[1 + \sum_{n=1}^{\infty} (z-1)^n \left(\frac{n}{n!} + \frac{1}{n!}\right)\big] \\
&= e\big[1 + \sum_{n=1}^{\infty} (z-1)^n \frac{n+1}{n!}\big].
\end{aligned}$$

21. (a) To obtain the partial fractions decomposition

$$\frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z},$$

we proceed in the usual way:

$$\begin{aligned}
\frac{1}{(1-z)(2-z)} &= \frac{A}{1-z} + \frac{B}{2-z} \\
&= \frac{A(2-z) + B(1-z)}{(1-z)(2-z)}; \\
1 &= A(2-z) + B(1-z) \\
\text{Take } z = 2 &\Rightarrow 1 = -B,\ B = -1. \\
\text{Take } z = 1 &\Rightarrow 1 = A.
\end{aligned}$$

Thus we obtain the desired partial fractions decomposition. Expanding each term in the partial fractions decomposition around $z_0 = 0$, we obtain

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1;$$

$$-\frac{1}{2-z} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n, \quad |\frac{z}{2}| < 1, \text{ or } |z| < 2.$$

So, for $|z| < 1$,

$$\frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z} = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n.$$

(b) We an derive the series in (a) by considering the Cauchy products of the series expansions of $\frac{1}{1-z}$ and $\frac{1}{2-z}$, as follows. From (a), we have

$$\frac{1}{1-z}\cdot\frac{1}{2-z}=\sum_{n=0}^{\infty}z^n\cdot\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}=\sum_{n=0}^{\infty}c_nz^n,$$

where c_n is obtained from the Cauchy product formula (see Exercise 21, Sec. 4.3):

$$\begin{aligned} c_n &= \sum_{k=0}^{n}a_kb_{n-k}, \\ a_k=1, \qquad & b_{n-k}=\frac{1}{2^{n-k+1}}, \\ c_n &= \sum_{k=0}^{n}\frac{1}{2^{n-k+1}}=\frac{1}{2^{n+1}}\sum_{k=0}^{n}2^k. \end{aligned}$$

(c) To show that the Cauchy product is the same as the series that we found in (a), we must prove that

$$\frac{1}{2^{n+1}}\sum_{k=0}^{n}2^k=\left(1-\frac{1}{2^{n+1}}\right).$$

But this is clear since

$$\sum_{k=0}^{n}2^k=1+2+2^2+\cdots+2^n=2^{n+1}-1,$$

and so

$$\frac{1}{2^{n+1}}\sum_{k=0}^{n}2^k=\frac{1}{2^{n+1}}(2^{n+1}-1)=\left(1-\frac{1}{2^{n+1}}\right).$$

The radius of the Maclaurin series is 1. This follows from our argument in (a) or from Theorem 1, since the function has a problem at $z=1$.

25. By the Exercise 24,

$$f(z)=-\frac{1}{(2i-z)^3}=-\frac{i}{8}\sum_{n=0}^{\infty}\binom{n+2}{2}\left(\frac{z}{2i}\right)^n$$

for all $|z|<2$.

29. (a) Since even functions are characterized by the fact that $f(z)-f(-z)=0$, and since

$$f(-z)=\sum_{n=0}^{\infty}c_n(-z)^n=\sum_{n=0}^{\infty}(-1)^nc_nz^n,$$

we have that f is even if and only if

$$f(z) - f(-z) = 0$$
$$\Updownarrow$$
$$\sum_{n=0}^{\infty} c_n z^n - \sum_{n=0}^{\infty} (-1)^n c_n z^n = \sum_{n=0}^{\infty} (1 - (-1)^n) c_n z^n = 0$$
$$\Updownarrow$$
$$(1 - (-1)^n) c_n = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$
$$\Updownarrow$$
$$2c_{2n+1} = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$
$$\Updownarrow$$
$$c_{2n+1} = 0 \quad \text{for all } n = 0, 1, 2, \ldots.$$

(b) Similarly, f is odd if and only if

$$f(z) + f(-z) = 0$$
$$\Updownarrow$$
$$\sum_{n=0}^{\infty} c_n z^n + \sum_{n=0}^{\infty} (-1)^n c_n z^n = \sum_{n=0}^{\infty} (1 + (-1)^n) c_n z^n = 0$$
$$\Updownarrow$$
$$(1 + (-1)^n) c_n = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$
$$\Updownarrow$$
$$2c_{2n} = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$
$$\Updownarrow$$
$$c_{2n} = 0 \quad \text{for all } n = 0, 1, 2, \ldots.$$

33. (a) The sequence of integers $\{l_n\}$ satisfies the recurrence relation $l_n = l_{n-1} + l_{n-2}$ for $n \geq 2$, with $l_0 = 1$ and $l_1 = 3$. As suggested, suppose that l_n occur as the Maclaurin coefficient of some analytic function $f(z) = \sum_{n=0}^{\infty} l_n z^n$, $|z| < R$. To derive the given identity for f, multiply the series by z and z^2, and then use the recurrence relation for the coefficients. Using $l_0 = 1$ and $l_1 = 3$, we obtain

$$f(z) = \sum_{n=0}^{\infty} l_n z^n = 1 + 3z + \sum_{n=2}^{\infty} l_n z^n;$$
$$z f(z) = \sum_{n=0}^{\infty} l_n z^{n+1} = \sum_{n=1}^{\infty} l_{n-1} z^n = l_0 z + \sum_{n=2}^{\infty} l_{n-1} z^n;$$
$$z f(z) = z + \sum_{n=2}^{\infty} l_{n-1} z^n;$$
$$z^2 f(z) = \sum_{n=0}^{\infty} l_n z^{n+2} = \sum_{n=2}^{\infty} l_{n-2} z^n.$$

Using the recurrence relation and the preceding identities, we obtain

$$\begin{aligned} f(z) &= 1+3z+\sum_{n=2}^{\infty} l_n z^n \\ &= 1+3z+\sum_{n=2}^{\infty}\big(l_{n-1}+l_{n-2}\big)z^n \\ &= 1+3z+\overbrace{\sum_{n=2}^{\infty} l_{n-1}z^n}^{zf(z)-z}+\overbrace{\sum_{n=2}^{\infty} l_{n-2}z^n}^{z^2 f(z)} \\ &= 1+3z+zf(z)-z+z^2 f(z) = 1+2z+zf(z)+z^2 f(z). \end{aligned}$$

Solving for $f(z)$, we obtain

$$f(z) = \frac{1+2z}{1-z-z^2}.$$

(b) To compute the Maclaurin series of f, we will use the result of Exercise 22:

$$\frac{1}{(z_1-z)(z_2-z)} = \frac{1}{z_1-z_2}\sum_{n=0}^{\infty}\frac{(z_1^{n+1}-z_2^{n+1})}{(z_1 z_2)^{n+1}}z^n, \quad |z|<|z_1|,\ z_1\neq z_2,\ |z_1|\leq|z_2|.$$

To derive this identity, start with the partial fractions decomposition

$$\frac{1}{(z_1-z)(z_2-z)} = \frac{1}{z_1-z_2}\left[\frac{1}{z_2-z}-\frac{1}{z_1-z}\right] = \frac{1}{z_1-z_2}\left[\frac{1}{z_2(1-\frac{z}{z_2})}-\frac{1}{z_1(1-\frac{z}{z_1})}\right].$$

Apply a geometric series expansion and simplify:

$$\begin{aligned} \frac{1}{(z_1-z)(z_2-z)} &= \frac{1}{z_1-z_2}\left[\frac{1}{z_2}\sum_{n=0}^{\infty}\left(\frac{z}{z_2}\right)^n-\frac{1}{z_1}\sum_{n=0}^{\infty}\left(\frac{z}{z_1}\right)^n\right] \\ &= \frac{1}{z_1-z_2}\sum_{n=0}^{\infty}\left(\frac{1}{z_2^{n+1}}-\frac{1}{z_1^{n+1}}\right)z^n \\ &= \frac{1}{z_1-z_2}\sum_{n=0}^{\infty}\frac{(z_1^{n+1}-z_2^{n+1})}{(z_1 z_2)^{n+1}}z^n. \end{aligned}$$

Now, consider the function

$$\frac{1}{1-z-z^2} = \frac{-1}{z^2+z-1} = \frac{-1}{(z_1-z)(z_2-z)},$$

where z_1 and z_2 are the roots of z^2+z-1:

$$z_1 = \frac{-1+\sqrt{5}}{2} \quad \text{and} \quad z_2 = \frac{-1-\sqrt{5}}{2},$$

arranged so that $z_1| < |z_2|$. These roots satisfy known relationships determined by the coefficients of the polynomial z^2+z-1. We will need the following easily verified identities:

$$z_1 - z_2 = \sqrt{5} \quad \text{and} \quad z_1 z_2 = -1.$$

We will also need the following identities:

$$\begin{aligned} z_1^n(z_1-2) &= (-1)^n \left(\frac{1-\sqrt{5}}{2}\right)^n \big(-2+\frac{-1+\sqrt{5}}{2}\big) \\ &= (-1)^n \left(\frac{1-\sqrt{5}}{2}\right)^n \big(\frac{-5+\sqrt{5}}{2}\big) \\ &= (-1)^n\sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} z_2^n(2-z_2) &= (-1)^n \left(\frac{1+\sqrt{5}}{2}\right)^n \big(2+\frac{1+\sqrt{5}}{2}\big) \\ &= (-1)^n \left(\frac{1+\sqrt{5}}{2}\right)^n \big(\frac{5+\sqrt{5}}{2}\big) \\ &= (-1)^n \left(\frac{1+\sqrt{5}}{2}\right)^n \sqrt{5}\big(\frac{1+\sqrt{5}}{2}\big) = (-1)^n\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

We are now ready to derive the desired Maclaurin series. We have

$$\begin{aligned} \frac{1}{1-z-z^2} &= \frac{-1}{z^2+z-1} = \frac{-1}{(z_1-z)(z_2-z)} \\ &= \frac{-1}{\sqrt{5}}\sum_{n=0}^\infty \frac{z_1^{n+1}-z_2^{n+1}}{(-1)^{n+1}} z^n = \frac{-1}{\sqrt{5}}\sum_{n=0}^\infty (-1)^{n+1}\big(z_1^{n+1}-z_2^{n+1}\big)z^n \\ &= \frac{-1}{\sqrt{5}}\overbrace{(z_1-z_2)}^{=\sqrt{5}}(-1) + \frac{-1}{\sqrt{5}}\sum_{n=1}^\infty (-1)^{n+1}\big(z_1^{n+1}-z_2^{n+1}\big)z^n \\ &= 1+\frac{1}{\sqrt{5}}\sum_{n=1}^\infty (-1)^n\big(z_1^{n+1}-z_2^{n+1}\big)z^n; \\ \frac{2z}{1-z-z^2} &= \frac{2}{\sqrt{5}}\sum_{n=0}^\infty (-1)^n\big(z_1^{n+1}-z_2^{n+1}\big)z^{n+1} = \frac{-2}{\sqrt{5}}\sum_{n=1}^\infty (-1)^n\big(z_1^n-z_2^n\big)z^n. \end{aligned}$$

So

$$\begin{aligned}
f(z) &= \frac{1+2z}{1-z-z^2} \\
&= 1+\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(-1)^n\big(2z_2^n-2z_1^n-z_2^{n+1}+z_1^{n+1}\big)z^n \\
&= 1+\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(-1)^n\big(z_2^n(2-z_2)+z_1^n(z_1-2)\big)z^n \\
&= 1+\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\left\{(-1)^n(-1)^n\sqrt{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]z^n\right\} \\
&= 1+\sum_{n=1}^{\infty}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]z^n.
\end{aligned}$$

Thus

$$l_n=\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, \quad n\geq 0.$$

37. We use the binomial series expansion from Exercise 36, with $\alpha=\frac{1}{2}$. Accordingly, for $|z|<1$,

$$(1+z)^{\frac{1}{2}}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}z^n,$$

where, for $n\geq 1$,

$$\begin{aligned}
\binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}=\frac{\frac{1}{2}\frac{-3}{2}\cdots\frac{-(2n-3)}{2}}{n!} \\
&= (-1)^{n-1}\frac{\frac{1}{2}\frac{3}{2}\cdots\frac{(2n-3)}{2}}{n!}=(-1)^{n-1}\frac{1}{2^n n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot 2\cdot 4\cdots(2n-2)}{2\cdot 4\cdots(2n-2)} \\
&= (-1)^{n-1}\frac{1}{2^n n!}\frac{(2n-2)!}{2\cdot 1\cdot 2\cdot 2\cdots 2\cdot(n-1)}=(-1)^{n-1}\frac{1}{2^n n!}\frac{(2n-2)!}{2^{n-1}\cdot 1\cdot 2\cdots(n-1)} \\
&= =(-1)^{n-1}\frac{1}{2^n n!}\frac{(2n-2)!}{2^{n-1}(n-1)!}=(-1)^{n-1}\frac{1}{2^n n!}\frac{(2n-2)!}{2^{n-1}(n-1)!}\frac{2n(2n-1)}{2n(2n-1)} \\
&= (-1)^{n-1}\frac{1}{2^n n!}\frac{(2n)!}{2^n n!(2n-1)}(-1)^{n-1}\frac{(2n)!(2n-1)}{2^{2n}(n!)^2}=(-1)^{n-1}\frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n}.
\end{aligned}$$

Thus

$$(1+z)^{\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n}z^n, \qquad |z|<1.$$

41. (a) and (b) There are several possible ways to derive the Taylor series expansion of $f(z)=\operatorname{Log} z$ about the point $z_0=-1+i$. Here is one way. Let $z_0=-1+i$, so $|z_0|=\sqrt{2}$. The function $\operatorname{Log} z$ is analytic except on the negative real axis and 0. So it is guaranteed by

Theorem 1 to have a series expansion in the largest disk around z_0 that does not intersect the negative real axis. Such a disc, as you can easily verify, has radius $\operatorname{Im} z_0 = 1$. However, as you will see shortly, the series that we obtain has a larger radius of convergence, namely $|z_0| = \sqrt{2}$ (of course, this is not a contradiction to Theorem 1).

Consider the function $\operatorname{Log} z$ in $B_1(z_0)$, where it is analytic. (The disk of radius 1, centered at z_0 is contained in the upper half-plane.) For $z \in B_1(z_0)$, we have $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$. Instead of computing the Taylor series of $\operatorname{Log} z$ directly, we will first compute the Taylor series of $\frac{1}{z}$, and then integrate term-by-term within the radius of convergence of the series (Theorem 3, Sec. 4.3). Getting ready to apply the geometric series result, we write

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{z_0 - (z_0 - z)} = \frac{1}{z_0\left(1 - \frac{z_0 - z}{z_0}\right)} \qquad (z_0 \neq 0)\\
&= \frac{1}{z_0} \cdot \frac{1}{1 - \frac{z_0 - z}{z_0}} = \frac{1}{z_0} \sum_{n=0}^{\infty} \left(\frac{z_0 - z}{z_0}\right)^n\\
&= \frac{1}{z_0} \sum_{n=0}^{\infty} (-1)^n \frac{\left(z - z_0\right)^n}{z_0^n},
\end{aligned}$$

where the series expansion holds for

$$\left|\frac{z_0 - z}{z_0}\right| < 1 \Leftrightarrow |z_0 - z| < |z_0|.$$

Thus the series representation holds in a disk of radius $|z_0| = \sqrt{2}$, around z_0. Within this disk, we can integrate the series term-by-term and get

$$\int_{z_0}^{z} \frac{1}{\zeta} d\zeta = \frac{1}{z_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^n} \int_{z_0}^{z} \left(\zeta - z_0\right)^n d\zeta = \frac{1}{z_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1) z_0^n} \left(z - z_0\right)^{n+1}.$$

Reindexing the series by changing $n+1$ to n, we obtain

$$\int_{z_0}^{z} \frac{1}{\zeta} d\zeta = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n z_0^n} \left(z - z_0\right)^n \qquad |z - z_0| < |z_0| = \sqrt{2}.$$

Now we have to decide what to write on the left side. The function $\operatorname{Log} z$ is an antiderivative of $\frac{1}{z}$ in the disk of radius 1, centered at z_0. (Remember that $\operatorname{Log} z$ is not analytic on the negative real axis, so we cannot take a larger disk.) So, for $|z - z_0| < 1$, we have

$$\int_{z_0}^{z} \frac{1}{\zeta} d\zeta = \operatorname{Log} \zeta \Big|_{z_0}^{z} = \operatorname{Log} z - \operatorname{Log} z_0.$$

Thus, for $|z - z_0| < 1$, we have

$$\operatorname{Log} z = \operatorname{Log} z_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n z_0^n} \left(z - z_0\right)^n,$$

even though the series on the right converges in the larger disk $|z - z_0| < \sqrt{2}$.

Solutions to Exercises 4.4

1. We have that

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}},$$

and this converges provided $\left|-\frac{1}{z}\right| < 1$, that is, when $1 < |z|$.

5. Since $\left|\frac{1}{z^2}\right| < 1$, we can use a geometric series in $\frac{1}{z^2}$ as follows. We have

$$\begin{aligned}\frac{1}{1+z^2} &= \frac{1}{z^2}\frac{1}{1-\frac{-1}{z^2}}\\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2(n+1)}}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^{2n}},\end{aligned}$$

where in the last series we shifted the index of summation by 1. Note that $(-1)^{n-1} = (-1)^{n+1}$, and so the two series that we derived are the same.

9. Since the function $f(z) := z + \dfrac{1}{z}$ can be written as:

$$z + \frac{1}{z} = 1 + (z-1) + \frac{1}{1+(z-1)} = 1 + (z-1) + \frac{1}{z-1}\frac{1}{1+\frac{1}{z-1}}.$$

Then the Laurent series expansion for $f(z)$ in the annulus $1 < |z-1|$ is:

$$z + \frac{1}{z} = 1 + (z-1) + \sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-1)^{n+1}},$$

where $1 < |z-1|$.

13. Since the function $f(z) := \dfrac{z}{(z+2)(z+3)}$ can be written as:

$$\frac{z}{(z+2)(z+3)} = \frac{-2}{z+2} + \frac{3}{z+3}.$$

(1) For the annulus $2 < |z|$:

$$\frac{1}{z+2} = \frac{1}{z}\frac{1}{1+\frac{2}{z}} = \sum_{n=0}^{\infty}(-1)^n 2^n\frac{1}{z^{n+1}}.$$

(2) For the annulus $|z| < 3$:

$$\frac{1}{z+3} = \frac{1}{3}\frac{1}{1+\frac{z}{3}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}z^n.$$

Then in the annulus $2 < |z| < 3$, we have the Laurent series for $f(z)$:

$$\frac{z}{(z+2)(z+3)} = \sum_{n=1}^{\infty}(-1)^n 2^n \frac{1}{z^n} + \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n} z^n,$$

where $2 < |z| < 3$.

17. First, derive the partial fractions decomposition

$$\frac{z^2 + (1-i)z + 2}{(z-i)(z+2)} = 1 + \frac{1}{z-i} - \frac{2}{z+2}.$$

The first step should be to reduce the degree of the numerator by dividing it by the denominator. As in Exercise 13, we handle each term separately, the constant term is to be left alone for now. In the annulus $1 < |z| < 2$, we have $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$. So to expand $\frac{1}{z-i}$, factor the z in the denominator and you'll get

$$\frac{1}{z-i} = \frac{1}{z(1-\frac{i}{z})} = \frac{1}{z}\frac{1}{1-\frac{i}{z}},$$

where $\left|\frac{i}{z}\right| < 1$ or $1 < |z|$. Apply a geometric series expansion: for $1 < |z|$,

$$\begin{aligned}\frac{1}{z-i} &= \frac{1}{z}\frac{1}{1-\frac{i}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty}\frac{i^{n-1}}{z^n}.\end{aligned}$$

To expand $\frac{2}{z+2}$, in the annulus $1 < |z| < 2$, because $\left|\frac{z}{2}\right| < 1$, we divide the denominator by 2 and get

$$\frac{2}{z+2} = \frac{2}{2(1+\frac{z}{2}} = \frac{1}{1-\frac{-z}{2}}.$$

Expand, using a geometric series, which is valid for $|z| < 2$, and get

$$\frac{2}{z+2} = \frac{1}{1-\frac{-z}{2}} = \sum_{n=0}^{\infty}\left(\frac{-z}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}.$$

Hence, for $1 < |z| < 2$,

$$1 + \frac{1}{z-i} - \frac{2}{z+2} = 1 - \sum_{n=1}^{\infty}\frac{i^{n-1}}{z^n} + \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}.$$

21. The function $f(z) = \dfrac{1}{(z-1)(z+i)}$ has isolated singularities at $z = 1$ and $z = -i$. If we start at the center $z_0 = -1$, the closest singularity is $-i$ and its distance to z_0 is $\sqrt{2}$. Thus $f(z)$ is analytic in the disk of radius $\sqrt{2}$ and center at $z_0 = -1$, which is the annulus $|z+1| < \sqrt{2}$. This is one of the Laurent series that we seek. Moving outside this disk, we encounter the second singularity at $z=1$. Thus $f(z)$ is analytic in the annulus $\sqrt{2}<|z+1|<2$, and has a Laurent series representation there. Finally, the function is analytic in the annulus $2<|z+1|$ and so ha a Laurent expansion there.

We now derive the three series expansions. Using a partial fractions decomposition, we have

$$f(z)=\frac{1}{(z-1)(z+i)}=\frac{A}{z-1}-\frac{A}{z+i},$$

where $A=\frac{1}{2}-\frac{i}{2}=\frac{1}{2}(1-i)$. We have, for $|z+1|<2$,

$$\begin{aligned}\frac{1}{z-1} &= \frac{1}{-2+(z+1)}=-\frac{1}{2}\frac{1}{1-\frac{z+1}{2}}\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n.\end{aligned}$$

For $|z+1|<\sqrt{2}$, we have $\left|\frac{z+1}{1-i}\right|<1$, and so

$$\begin{aligned}\frac{1}{z+i} &= \frac{1}{(i-1)+(z+1)}=\frac{-1}{1-i}\frac{1}{1-\frac{z+1}{1-i}}\\ &= \frac{-1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z+1}{1-i}\right)^n.\end{aligned}$$

Thus, for $|z+1|<\sqrt{2}$, we have

$$\begin{aligned}f(z) &= \frac{1}{(z-1)(z+i)}=\frac{A}{z-1}-\frac{A}{z+i}\\ &= -\frac{A}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n+\frac{A}{1-i}\sum_{n=0}^{\infty}\left(\frac{z+1}{1-i}\right)^n\\ &= -\frac{1-i}{4}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{1-i}\right)^n.\end{aligned}$$

For $\sqrt{2}<|z+1|$, we have $\left|\frac{1-i}{z+1}\right|<1$, and so

$$\begin{aligned}\frac{1}{z+i} &= \frac{1}{(i-1)+(z+1)}=\frac{1}{z+1}\frac{1}{1-\frac{1-i}{z+1}}\\ &= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1-i}{z+1}\right)^n=\sum_{n=0}^{\infty}\frac{(1-i)^n}{(z+1)^{n+1}}\\ &= \frac{1}{1-i}\sum_{n=1}^{\infty}\frac{(1-i)^n}{(z+1)^n}.\end{aligned}$$

So, if $\sqrt{2}<|z+1|<2$, then

$$\begin{aligned}f(z) &= \frac{1}{(z-1)(z+i)}=\frac{A}{z-1}-\frac{A}{z+i}\\ &= -\frac{1-i}{4}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n-\frac{1-i}{2}\sum_{n=0}^{\infty}\frac{(1-i)^n}{(z+1)^{n+1}}.\end{aligned}$$

Finally, for $2 < |z+1|$, we have

$$\begin{aligned}
\frac{1}{z-1} &= \frac{1}{-2+(z+1)} = \frac{1}{z+1}\frac{1}{1-\frac{2}{z+1}} \\
&= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^n}{(z+1)^n}.
\end{aligned}$$

So, if $2 < |z+1|$, then

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)(z+i)} = \frac{A}{z-1} - \frac{A}{z+i} \\
&= \frac{1-i}{2}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}} - \frac{1-i}{2}\sum_{n=0}^{\infty}\frac{(1-i)^n}{(z+1)^{n+1}} \\
&= \frac{1-i}{4}\sum_{n=1}^{\infty}\frac{2^n}{(z+1)^n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(1-i)^n}{(z+1)^n}.
\end{aligned}$$

25. In this problem, the idea is to evaluate the integral by integrating a Laurent series term-by-term. This process is justified by Theorem 1, which asserts that the Laurent series converges absolutely and uniformly on any closed and bounded subset of its domain of convergence. Since a path is closed and bounded, if the path lies in the domain of convergence of the Laurent series, then the series converges uniformly on the path. Hence, by Corollary 1, Sec. 4.2, the series can be differentiated term-by-term. We now present the details of the solution. Using the Maclaurin series of $\sin z$, we have for all $z \neq 0$,

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-(2n+1)}.$$

Thus

$$\int_{C_1(0)}\sin\frac{1}{z}\,dz = \int_{C_1(0)}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-(2n+1)}\right)dz = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_{C_1(0)}z^{-(2n+1)}\,dz.$$

We now recall the important integral formula: for any integer m:

$$\int_C z^m\,dz = \begin{cases} 2\pi i & \text{if } m=-1, \\ 0 & \text{otherwise,}\end{cases}$$

where C is any positively oriented simple closed path containing 0 (see Example 4, Sec. 3.4). Thus,

$$\int_{C_1(0)} z^{-(2n+1)}\,dz = \begin{cases} 2\pi i & \text{if } n=0, \\ 0 & \text{otherwise.}\end{cases}$$

Hence all the terms in the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_{C_1(0)}z^{-(2n+1)}\,dz$$

are 0, except the term that corresponds to $n = 0$, which is equal to $2\pi i$. So

$$\int_{C_1(0)} \sin\frac{1}{z}\,dz = 2\pi i.$$

29. We follow the same strategy as in Exercise 25 and use the series expansion from Exercise 5. We have

$$\begin{aligned}\int_{C_4(0)} \operatorname{Log}\left(1+\frac{1}{z}\right) dz &= \int_{C_4(0)} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\frac{1}{z^n}\right) dz\\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{C_4(0)} \frac{1}{z^n} dz\\ &= 2\pi i,\end{aligned}$$

where we have used the fact that $\int_{C_4(0)} \frac{1}{z^n} dz = 2\pi i$ if $n = 1$ and 0 otherwise.

33. (a) Parametrizing $\zeta(\theta) = e^{i\theta}, -\pi \le \theta \le \pi$, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_1(0)} e^{\frac{z}{2}\left(\zeta-\frac{1}{\zeta}\right)} \frac{d\zeta}{\zeta^{n+1}} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{z}{2}\left(e^{i\theta}-e^{-i\theta}\right)} e^{-in\theta}\,d\theta\\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta} e^{-in\theta}\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z\sin\theta-n\theta)}\,d\theta\\ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\cos(z\sin\theta-n\theta) &+ i\sin(z\sin\theta-n\theta)\Big)\,d\theta.\end{aligned}$$

Now, exploiting symmetry, note that $z\sin\theta - n\theta$ is odd, so $\sin(z\sin\theta - n\theta)$ is also odd and that portion of the integral vanishes. Furthermore, $\cos(z\sin\theta - n\theta)$ is even, so we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\cos(z\sin\theta-n\theta) + i\sin(z\sin\theta-n\theta)\Big)\,d\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(z\sin\theta-n\theta)\,d\theta = \frac{1}{\pi}\int_{0}^{\pi}\cos(z\sin\theta-n\theta)\,d\theta.\end{aligned}$$

(b) If $z = x$ is real, we have that $|\cos(x\sin\theta - n\theta)| \le 1$ for all θ, so by the ML-inequality, we have

$$|J_n(x)| = \frac{1}{\pi}\left|\int_0^{\pi}\cos(z\sin\theta-n\theta)\,d\theta\right| \le \frac{1}{\pi}(1)(\pi) = 1.$$

Solutions to Exercises 4.5

1. Since

$$(1-z^2)\sin z = (1-z)(1+z)\sin z$$

and all zeros of $\sin z$ are of the form $k\pi$ for $k \in \mathbb{Z}$, we have that the zeros of this function are at $1, -1, k\pi$ for $k \in \mathbb{Z}$. By the expansion of $\sin z$ around zero, we have that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)$$

and the zero of $\sin z$ at zero has order one. Similarly, by symmetry of $\sin z$, all zeros of $\sin z$ have order one. We also see that the zeros -1 and 1 of the original function have order one, so all zeros have order one.

5. Since all zeros of $\sin z$ are isolated, which are $k\pi$, $k \in \mathbb{Z}$:

(1) At $z = 0$, we have the Taylor expansion:

$$\sin z = z\Big(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\Big) =: z\lambda(z),$$

where $\lambda(0) \neq 0$. Therefore we have

$$\frac{\sin^7 z}{z^4} = \frac{z^7\lambda^7(z)}{z^4} = z^3\lambda^7(z).$$

where $\lambda^7(0) \neq 0$. Then $\dfrac{\sin^7 z}{z^4}$ has zero of order 3 at 0.

(2) At $z = k\pi, k \in \mathbb{Z}\backslash\{0\}$, we have

$$\frac{\sin^7 z}{z^4} = z^7\frac{\lambda^7(z)}{z^4}, \qquad \frac{\lambda^7(z)}{z^4} \neq 0.$$

Then $\dfrac{\sin^7(z)}{z^4}$ has zero of order 7 at $k\pi, k \in \mathbb{Z}\backslash\{0\}$.

9. Let $f(z) := 1 - \dfrac{z^2}{2} - \cos z$, and we know $f(z)$ is entire. Then the Taylor expansion for $f(z)$ at 0 is:

$$\begin{aligned} f(z) &= 1 - \frac{z^2}{2} - \cos z = 1 - \frac{z^2}{2} - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) \\ &= -\frac{z^4}{4!} + \frac{z^6}{6!} - \cdots = z^4\Big(-\frac{1}{4!} + \frac{z^2}{6!} - \cdots\Big) \\ &=: z^4\lambda(z). \end{aligned}$$

Since $\lambda(z) = -\dfrac{1}{4!} + \dfrac{z^2}{6!} - \cdots$ is a power series which converges for all z, then $\lambda(z)$ is entire. And $\lambda(0) = -\dfrac{1}{4!} \neq 0$. Therefore, $f(z) = 1 - \dfrac{z^2}{2} - \cos z$ has the zero of order 4 at $z_0 = 0$.

13. Clearly, the function

$$f(z) = \frac{1-z^2}{\sin z} + \frac{z-1}{z+1}$$

has isolated singularities at -1 and $k\pi$, where k is an integer. These singularities are all simple poles. To prove the last assertion, it is easier to work with each part of the function separately. First, show that $\frac{1-z^2}{\sin z}$ has a simple pole at the zeros of $\sin z$, which follows immediately from the fact that the zeros of $\sin z$ are simple zeros. Second, show that $\frac{z-1}{z+1}$ has a simple pole at $z = -1$, which follows immediately from the fact that -1 is a simple zero of $z + 1$. Now to put the two terms together, you can use the following fact:

If $f(z)$ has a pole of order m at z_0 and $g(z)$ is analytic at z_0, then $f(z) + g(z)$ has a pole of order m at z_0.

This result is easy to prove using, for example, Theorem 8.

17. Write

$$z \tan \frac{1}{z} = z \frac{\sin \frac{1}{z}}{\cos \frac{1}{z}}.$$

The problem points of this function are at 0 and at the zeros of the equation $\cos \frac{1}{z} = 0$. Solving, we find

$$\frac{1}{z} = \frac{\pi}{2} + k\pi \Rightarrow z = z_k = \frac{2}{\pi(2k+1)}, \quad k \text{ an integer.}$$

Since, as $k \to \infty$, $z_k \to 0$, the function $f(z)$ is not analytic in any punctured disk of the form $0 < |z|$. Thus 0 is not an isolated singularity. At all the other points z_k, the singularity is isolated and the order of the singularity is equal to the order of the zero of $\cos z$ at z_k. Since the zeros of $\cos z$ are all simple (this is very similar to Example 1), we conclude that $f(z)$ has simple poles at z_k.

21. The function $f(z) := \dfrac{1}{z} - \sin \dfrac{1}{z}$ is analytic when $z \neq 0$, then $z = 0$ is the isolated singularity of $f(z)$. Thus the Laurent series expansion for $f(z)$ about 0 can be written as:

$$f(z) = z^{-1} - \left(z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \cdots \right) = \frac{z^{-3}}{3!} - \frac{z^{-5}}{5!} + \frac{z^{-7}}{7!} - \cdots .$$

Since $a_n \neq 0$ for infinitely many $n < 0$, therefore $z = 0$ is an essential singularity of $f(z)$.

25. Determining the type of singularity of $f(z) = \frac{1}{z+1}$ at ∞ is equivalent to determining the type of singularity of

$$f\left(\frac{1}{z}\right) = \frac{1}{\frac{1}{z} + 1} = \frac{z}{1+z}$$

at $z = 0$. Since $f\left(\frac{1}{z}\right)$ has a removable singularity at 0, we conclude that f has a removable singularity at $z = \infty$. Note that this is consistent with our characterization of singularities according to the behavior of the function at the point. Since $f(z) \to 0$ as $z \to \infty$, we conclude that f has a removable singularity and may be redefined to have a zero at ∞.

29. In order to determine the type of singularity of $f(z) := \sin \dfrac{1}{z}$ at ∞, it is equivalent to determine the type of singularity of $f\left(\dfrac{1}{z}\right) = \sin z$ at 0. It is clear that $f\left(\dfrac{1}{z}\right)$ has a removable singularity at 0, since

$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = \lim_{z\to 0} \sin z = 0.$$

Therefore, $f(z)$ has a removable singularity at ∞. And since $f(z) \to 0$, as $z \to \infty$, the function has a zero at ∞.

33. (a) Suppose that f is entire and bounded. Consider $g(z) = f(\frac{1}{z})$. Then g is analytic at all $z \neq 0$. So $z = 0$ is an isolated singularity of $g(z)$. For all $z \neq 0$, we have $|g(z)| = |f(\frac{1}{z})| \leq M < \infty$, where M is a bound for $|f(z)|$, which is supposed to exist. Consequently, $g(z)$ is bounded around 0 and so 0 is a removable singularity of $g(z)$.
(b) Since f is entire, it has a Maclaurin series that converges for all z. Thus, for all z, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. In particular, we can evaluate this series at $\frac{1}{z}$ and get, for $z \neq 0$,

$$g(z) = f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}.$$

By the uniqueness of Laurent series expansion, it follows that this series is the Laurent series of g. But g has a removable singularity at 0. So all the terms with negative powers of z must be zero, implying that $g(z) = a_0$ and hence $f(z) = a_0$ is a constant.

37. (a) If f has a pole of order $m \geq 1$ at z_0, then

$$f(z) = \frac{1}{(z - z_0)^m}\phi(z),$$

where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$. (See (6), Sec. 4.6.) So if n is a positive integer, then

$$[f(z)]^n = \frac{1}{(z - z_0)^{mn}}\phi^n(z) = \frac{1}{(z - z_0)^{mn}}\psi(z),$$

where ψ is analytic at z_0 and $\psi(z_0) \neq 0$. Thus $[f(z)]^n$ has a pole at z_0 of order mn if $n > 0$. If $n < 0$, then

$$[f(z)]^n = (z - z_0)^{-mn}\frac{1}{\phi^n(z)} = (z - z_0)^{-mn}\psi(z),$$

where ψ is analytic at z_0 and $\psi(z_0) \neq 0$. Thus $[f(z)]^n$ has a zero at z_0 of order $-mn$ if $n < 0$.
(b) If f has an essential singularity at z_0 then $|f(z)|$ is neither bounded nor tends to infinity at z_0. Clearly, the same holds for $|[f(z)]^n| = |f(z)|^n$: It is neither bounded nor tends to ∞ near z_0. Thus $[f(z)]^n$ has an essential singularity at z_0.

41. Let f be such a function. Since f is entire, f is continuous on $\mathbb{C}$, so

$$f(0) = f\left(\lim_{n\to\infty} \frac{1}{n}\right) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} 0 = 0$$

and we have that $z_0 = 0$ is a zero of f. Furthermore, this zero is not isolated, since any neighborhood of zero contains a number of the form $\frac{1}{n}$ for sufficiently large n. By Theorem 4.5.4, since f is analytic on $\mathbb{C}$ and has a non-isolated zero, f vanishes everywhere on $\mathbb{C}$. Thus, the only such function is $f(z) = 0$ for all $z \in \mathbb{C}$.

Solutions to Exercises 4.6

1. No: by the Schwarz-Pick Theorem, for any analytic function g from the unit disc to itself, we must have, for all a in the unit disc,

$$\left|g'(a)\right| \leq \frac{1-|g(a)|^2}{1-|a|^2},$$

but this would imply that $g'(\frac{1}{5}) \leq \frac{1-\frac{16}{25}}{1-\frac{1}{25}} = \frac{3}{8}$, and $\frac{5}{12} \not\leq \frac{3}{8}$.

5. By Schwarz's lemma, we have that $|f'(0)| \leq 1$ and $|(f^{-1})'(0)| = \frac{1}{|f'(0)|} \leq 1$, so $|f'(0)| = 1$. Since this equality holds, Schwarz's lemma gives that $f(z) = cz$ with $|c| \leq 1$. But then $f^{-1}(z) = \frac{1}{c}z$, so we also have that $\left|\frac{1}{c}\right| \leq 1$, so $|c| = 1$.

9. If p has no zeros inside the unit disk, then by Corollary 3.9.10, $p(z) = A$ is constant. Since p has modulus 1 on the unit circle, we have that $|A| = 1$.

Now suppose that p has zeros $a_1, a_2, \ldots, a_m$ inside the unit circle, counted according to multiplicity. Then we can write $p(z) = (a_1 - z)(a_2 - z)\cdots(a_m - z)q(z)$, where q is a polynomial with no zeros inside the unit circle. Multiplying p by a product of linear transformations of the form $\phi_{a_j}(z) = \frac{1-\overline{a_j}z}{a_j - z}$ gives us

$$\begin{aligned} F(z) = p(z)\prod_{j=1}^{m}\phi_{a_j}(z) &= (a_1 - z)(a_2 - z)\cdots(a_m - z)q(z)\frac{1-\overline{a_1}z}{a_1 - z}\frac{1-\overline{a_2}z}{a_2 - z}\cdots\frac{1-\overline{a_m}z}{a_m - z} \\ &= q(z)(1-\overline{a_1}z)(1-\overline{a_2}z)\cdots(1-\overline{a_m}z). \end{aligned}$$

Then F is a polynomial with no zeros inside the unit circle, and since each ϕ_{a_j} has modulus 1 on the unit circle, F also has modulus on the unit circle. Thus, by Corollary 3.9.10, $F(z) = A$ is a constant, and we have $|A| = 1$. But since F is a constant polynomial, we must have that each $a_j = 0$, and $q(z) = A$ is a constant. Thus, $p(z) = (-1)^m A z^m = A' z^m$, where $A' = (-1)^m A$ and $|A'| = 1$.

13. No: apply Schwarz's lemma to $g(z) = \frac{f(z)}{3}$ to yield that $|g'(0)| \leq 1$, that is, $|f'(0)| \leq 3$.

Solutions to Exercises 5.1

1. We write

$$f(z)=\frac{1+z}{z}=\frac{1}{z}+1.$$

Then it is clear that f has one simpe pole at $z_0=0$. The Laurent series expansion of $f(z)$ at $z_0=0$ is already given. The residue at 0 is the coefficient of $\frac{1}{z}$ in the Laurent series a_{-1}. Thus $a_{-1}=\operatorname{Res}(f,\,0)=1$.

.

5. We have one pole of order 3 at $z_0=-3i$. We write

$$\begin{aligned}
\left(\frac{z-1}{z+3i}\right)^3 &= \frac{1}{(z+3i)^3}\left[(z+3i)+(-3i-1)\right]^3\\
&= \frac{1}{(z+3i)^3}\left[(z+3i)^3+3(z+3i)^2(-1-3i)+3(z+3i)(-1-3i)+(-3i-1)^3\right]\\
&= 1+3\frac{(-1-3i)}{z+3i}+3\frac{(-1-3i)}{(z+3i)^2}+\frac{(-3i-1)^3}{(z+3i)^3}.
\end{aligned}$$

Thus

$$a_{-1}=3(-1-3i)=\operatorname{Res}\left(\left(\frac{z-1}{z+3i}\right)^3,-3i\right).$$

9. Write

$$f(z)=\csc(\pi z)\,\frac{z+1}{z-1}=\frac{1}{\sin\pi z}\frac{z+1}{z-1}.$$

Simple poles at the integers, $z=k$, $z\neq 1$. For $k\neq 1$,

$$\begin{aligned}
\operatorname{Res}(f,k) &= \lim_{z\to k}(z-k)\frac{1}{\sin\pi z}\frac{z+1}{z-1}\\
&= \lim_{z\to k}\frac{z+1}{z-1}\lim_{z\to k}\frac{(z-k)}{\sin\pi z}\\
&= \frac{k+1}{k-1}\lim_{z\to k}\frac{1}{\pi\cos\pi z} \qquad \text{(L'Hospital's rule)}\\
&= \frac{k+1}{k-1}\frac{1}{\pi\cos k\pi}=\frac{(-1)^k}{\pi}\frac{k+1}{k-1}.
\end{aligned}$$

At $z_0=1$, we have a pole of order 2. To simplify the computation of the residue, let's rewrite $f(z)$ as follows:

$$\begin{aligned}
\frac{1}{\sin\pi z}\frac{z+1}{z-1} &= \frac{1}{\sin\pi z}\frac{(z-1)+2}{z-1}\\
&= \frac{1}{\sin\pi z}+\frac{2}{(z-1)\sin\pi z}
\end{aligned}$$

We have

$$\operatorname{Res}(f,1)=\operatorname{Res}\left(\frac{1}{\sin\pi z},\ 1\right)+\operatorname{Res}\left(\frac{2}{(z-1)\sin\pi z},\ 1\right);$$

$$\operatorname{Res}\left(\frac{1}{\sin \pi z}, 1\right) = \lim_{z \to 1}(z-1)\frac{1}{\sin \pi z} = -\frac{1}{\pi} \qquad \text{(Use l'Hospital's rule.);}$$

$$\begin{aligned}
\operatorname{Res}\left(\frac{2}{(z-1)\sin \pi z}, 1\right) &= \lim_{z \to 1} \frac{d}{dz}\frac{2(z-1)}{\sin \pi z} \\
&= 2\lim_{z \to 1} \frac{\sin \pi z - (z-1)\pi \cos \pi z}{(\sin \pi z)^2} \\
&= 2\lim_{z \to 1} \frac{\pi \cos \pi z - \pi \cos \pi z + (z-1)\pi \sin \pi z}{2\pi \sin \pi z \cos \pi z} = 0.
\end{aligned}$$

So $\operatorname{Res}(f, 1) = -\frac{1}{\pi}$.

13. The easiest way to compute the integral is to apply Cauchy's generalized formula with

$$f(z) = \frac{z^2 + 3z - 1}{z^2 - 3},$$

which is analytic inside and on $C_1(0)$. Hence

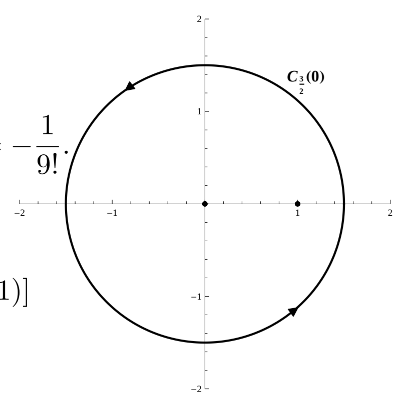
$$\int_{C_1(0)} \frac{z^2 + 3z - 1}{z(z^2 - 3)}\, dz = 2\pi i f(0) = 2\pi i\left(\frac{1}{3}\right) = \frac{2\pi i}{3}.$$

Note that from this value, we conclude that

$$\operatorname{Res}\left(\frac{z^2 + 3z - 1}{z(z^2 - 3)},\ 0\right) = \frac{1}{3}$$

because the integral is equal to $2\pi i$ times the residue at 0.

17. The function Let $f(z) = \dfrac{1}{z(z-1)(z-2)\cdots(z-10)}$, then $z = 0$ and $z = 1$ are simple poles of $f(z)$ inside $C_{\frac{3}{2}}(0)$. Since $C_{\frac{3}{2}}(0)$ is the circle positively oriented centered at the origin with radius $\dfrac{3}{2}$, with



$$\operatorname{Res}(f(z), 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{1}{(z-1)\cdots(z-10)} = \frac{1}{10!},$$

$$\operatorname{Res}(f(z), 1) = \lim_{z \to 1}(z-1) f(z) = \lim_{z \to 1} \frac{1}{z(z-2)\cdots(z-10)} = -\frac{1}{9!}.$$

Then the path integral can be evaluated by

$$\begin{aligned}
\int_{C_{\frac{3}{2}}(0)} \frac{\mathrm{d}z}{z(z-1)\cdots(z-10)} &= 2\pi i[\operatorname{Res}(f(z), 0) + \operatorname{Res}(f(z), 1)] \\
&= 2\pi i\left(\frac{1}{10!} - \frac{1}{9!}\right) = -\frac{18\pi i}{10!}.
\end{aligned}$$

Therefore, the path integral $\displaystyle\int_{C_{\frac{3}{2}}(0)} \frac{\mathrm{d}z}{z(z-1)(z-2)\cdots(z-10)} = -\frac{18\pi i}{10!}$.

21. The function

$$f(z) = \frac{e^{z^2}}{z^6}$$

has a pole of order 6 at 0. To compute the residue at 0, we find the coefficient a_1 in the Laurent series expansion about 0. We have

$$\frac{1}{z^6} e^{z^2} = \frac{1}{z^6} \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!}.$$

It is clear that this expansion has no terms with odd powers of z, positive or negative. Hence $a_{-1} = 0$ and so

$$\int_{C_1(0)} \frac{e^{z^2}}{z^6}\, dz = 2\pi i \operatorname{Res}(0) = 0.$$

25. Same approach as in Exercise 21:

$$\begin{aligned} \frac{\sin z}{z^6} &= \frac{1}{z^6} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= \frac{1}{z^6} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right] \end{aligned}$$

Coefficient of $\frac{1}{z}$: $a_{-1} = \frac{1}{5!}$, so

$$\int_{C_1(0)} \frac{\sin z}{z^6}\, dz = 2\pi i \operatorname{Res}(0) = \frac{2\pi i}{5!}.$$

29. (a) The Order of a pole of $\csc(\pi z) = \frac{1}{\sin \pi z}$ is the order of the zero of

$$\frac{1}{\csc(\pi z)} = \sin \pi z.$$

Since the zeros of $\sin \pi z$ occur at the integers and are all simple zeros (see Example 1, Section 4.6), it follows that $\csc \pi z$ has simple poles at the integers.
(b) For an integer k,

$$\begin{aligned} \operatorname{Res}\big(\csc \pi z,\, k\big) &= \lim_{z \to k} (z-k) \csc \pi z = \lim_{z \to k} \frac{z-k}{\sin \pi z} \\ &= \lim_{z \to k} \frac{1}{\pi \cos \pi z} \qquad \text{(l'Hospital's rule)} \\ &= \frac{(-1)^k}{\pi}. \end{aligned}$$

(c) Suppose that f is analytic at an integer k. Apply Proposition 1(iii), then

$$\operatorname{Res}\big(f(z) \csc(\pi z),\, k\big) = \frac{(-1)^k}{\pi} f(k).$$

33. A Laurent series converges absolutely in its annulus of convergence. Thus to multiply two Laurent series, we can use Cauchy products and sum the terms in any order. Write

$$\begin{aligned} f(z)g(z) &= \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n \\ &= \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \end{aligned}$$

where c_n s obtained by collecting all the terms in $(z-z_0)^n$, after expanding the product. Thus

$$c_n = \sum_{j=-\infty}^{\infty} a_j b_{n-j};$$

in particular

$$c_{-1} = \sum_{j=-\infty}^{\infty} a_j b_{-1-j},$$

and hence

$$\operatorname{Res}\big(f(z)g(z),\, z_0\big) = \sum_{j=-\infty}^{\infty} a_j b_{-j-1}.$$

37. (a) We have, Exercise 35(a), Section 4.5,

$$J_0(z) = \frac{1}{2\pi i}\int_{C_1(0)} e^{\frac{z}{2}\left(\zeta - \frac{1}{\zeta}\right)}\,\frac{d\zeta}{\zeta}.$$

Thus

$$\int_0^\infty J_0(t)e^{-st}\,dt = \frac{1}{2\pi i}\int_0^\infty \int_{C_1(0)} e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\,\frac{d\zeta}{\zeta}\,dt$$

(b) For ζ on $C_1(0)$, we have

$$\zeta - \frac{1}{\zeta} = \zeta - \overline{\zeta} = 2i\,\mathrm{Im}\,(\zeta),$$

which is 0 if $\mathrm{Im}\,(\zeta) = 0$ (i.e., $\zeta = \pm 1$) or is purely imaginary. In any case, for all $\zeta \in C_1(0)$, and all real $s > 0$ and t, we have

$$\left| e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)} \right| = \left|e^{-ts}\right| \left| e^{\frac{t}{2}(\zeta-\frac{1}{\zeta})} \right| = e^{-ts}.$$

So, by the inequality on integrals (Th.2, Sec. 3.2),

$$\begin{aligned} \left| \frac{1}{2\pi i}\int_{C_1(0)} e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\,\frac{d\zeta}{\zeta} \right| &\le \frac{2\pi}{2\pi}\max_{\zeta\in C_1(0)} \left| e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\,\frac{1}{\zeta} \right| \\ &= \max_{\zeta\in C_1(0)} \left| e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)} \right| \qquad (|\zeta| = 1) \\ &\le e^{-ts} \end{aligned}$$

Thus the iterated integral in (a) is absolutely convergent because

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_0^\infty \int_{C_1(0)} e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\frac{d\zeta}{\zeta}\,dt\right| &\leq \int_0^\infty \left|\int_{C_1(0)} e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\frac{d\zeta}{\zeta}\right|\,dt\\ &\leq \int_0^\infty e^{-ts}\,dt = \frac{1}{s} < \infty\end{aligned}$$

(c) Interchange the order of integration, and evaluate the integral in t, and get

$$\begin{aligned}\int_0^\infty J_0(t)e^{-st}\,dt &= \frac{1}{2\pi i}\int_{C_1(0)}\int_0^\infty e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\,dt\frac{d\zeta}{\zeta}\\ &= \frac{1}{2\pi i}\int_{C_1(0)} \frac{1}{\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\Big|_0^\infty \frac{d\zeta}{\zeta}\\ &= \frac{1}{\pi i}\int_{C_1(0)}\frac{1}{-\zeta^2+2s\zeta+1}d\zeta\end{aligned}$$

because, as $t\to\infty$,

$$\left|e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\right| = e^{-ts}\to 0.$$

(d) We evaluate the integral using the residue theorem. We have simple poles at

$$\zeta = \frac{-s\pm\sqrt{s^2+1}}{-1} = s\pm\sqrt{s^2+1}.$$

Only $s-\sqrt{s^2+1}$ is inside $C_0(1)$. To see this, note that because $s>0$ and $\sqrt{s^2+1}>1$, we have $s+\sqrt{s^2+1}>1$. Also

$$s<\sqrt{s^2+1}<1+s \quad\Rightarrow\quad -1-s<-\sqrt{s^2+1}<-s \quad\Rightarrow\quad -1<s-\sqrt{s^2+1}<0.$$

By Proposition 5.1.3(ii)

$$\begin{aligned}\operatorname{Res}\left(\frac{1}{-\zeta^2+2s\zeta+1}, s-\sqrt{s^2+1}\right) &= \frac{1}{-2\zeta+2s}\Big|_{\zeta=s-\sqrt{s^2+1}}\\ &= \frac{1}{2\sqrt{s^2+1}}\end{aligned}$$

Thus, for $s>0$,

$$\int_0^\infty J_0(t)e^{-st}\,dt = \frac{2\pi i}{\pi i}\frac{1}{2\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}}.$$

(e) Repeat the above steps making the appropriate changes.
Step (a'): Use the integral representation of J_n and get

$$I_n = \int_0^\infty J_n(t)e^{-st}\,dt = \frac{1}{2\pi i}\int_0^\infty\int_{C_1(0)} e^{-t\left(s-\frac{1}{2}(\zeta-\frac{1}{\zeta})\right)}\frac{d\zeta}{\zeta^{n+1}}\,dt$$

Step (b') is exactly like (b) because, for ζ on $C_1(0)$, we have $|\zeta^{n+1}| = |\zeta| = 1$.
Step (c'): As in (c), we obtain

$$I_n = \frac{1}{\pi i}\int_{C_1(0)} \frac{1}{(-\zeta^2+2s\zeta+1)\zeta^n}d\zeta.$$

Let $\eta = \frac{1}{\zeta} = \overline{\zeta}$. Then $d\zeta = -\frac{1}{\eta^2}d\eta$. As ζ runs through $C_1(0)$ in the positive direction, η runs through $C_1(0)$ in the negative direction. Hence

$$I_n = \frac{1}{\pi i}\int_{-C_1(0)} \frac{-\frac{1}{\eta^2}d\eta}{(-\frac{1}{\eta^2}+2s\frac{1}{\eta}+1)\frac{1}{\eta^n}} = \frac{1}{\pi i}\int_{C_1(0)} \frac{\eta^n d\eta}{\eta^2+2s\eta-1}.$$

Step (d'): We evaluate the integral using the residue theorem. We have simple poles at

$$\zeta = \frac{-s\pm\sqrt{s^2+1}}{-1} = -s\pm\sqrt{s^2+1}.$$

Only $-s+\sqrt{s^2+1}$ is inside $C_0(1)$. (Just argue as in (d).) By Proposition 1(ii)

$$\begin{aligned}\operatorname{Res}\left(\frac{\eta^n}{\eta^2+2s\eta-1}, -s+\sqrt{s^2+1}\right) &= \left.\frac{\eta^n}{2\eta+2s}\right|_{\eta=-s+\sqrt{s^2+1}} \\ &= \frac{\left(\sqrt{s^2+1}-s\right)^n}{2\sqrt{s^2+1}}\end{aligned}$$

Thus, for $s>0$,

$$\int_0^\infty J_n(t)e^{-st}\,dt = \frac{1}{\sqrt{s^2+1}}\left(\sqrt{s^2+1}-s\right)^n.$$

Solutions to Exercises 5.2

1. Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $d\theta = \frac{-i}{z}\,dz$, $\cos\theta = \frac{z+1/z}{2}$. Then

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} &= \int_{C_1(0)} \frac{-\frac{i}{z}dz}{2-\frac{(z+1/z)}{2}} \\
&= -i\int_{C_1(0)} \frac{dz}{2z-\frac{z^2}{2}-\frac{1}{2}} \\
&= -i\int_{C_1(0)} \frac{dz}{-\frac{z^2}{2}+2z-\frac{1}{2}} \\
&= 2\pi \sum_j \operatorname{Res}\left(\frac{1}{-\frac{z^2}{2}+2z-\frac{1}{2}},\, z_j\right),
\end{aligned}$$

where the sum of the residues extends over all the poles of $\frac{1}{2z-\frac{z^2}{2}-\frac{1}{2}}$ inside the unit disk. We have

$$-\frac{z^2}{2}+2z-\frac{1}{2}=0 \quad\Leftrightarrow\quad z^2-4z+1=0.$$

The roots are $z = 2\pm\sqrt{3}$, and only $z_1 = 2-\sqrt{3}$ is inside $C_1(0)$. We compute the residue using Proposition 1(ii), Sec. 5.1:

$$\operatorname{Res}\left((-\frac{z^2}{2}+2z-\frac{1}{2})^{-1},\, z_1\right) = \frac{1}{-z_1+2} = \frac{1}{\sqrt{3}}.$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

5. Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $d\theta = \frac{-i\,dz}{z}$, $\cos\theta = \frac{z+1/z}{2}$, and $\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^2+\frac{1}{z^2}}{2}$. Then

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta &= -i\int_{C_1(0)} \frac{\frac{1}{2}(z^2+\frac{1}{z^2})}{5+4\frac{(z+1/z)}{2}}\,\frac{dz}{z} \\
&= -\frac{i}{2}\int_{C_1(0)} \frac{z^4+1}{5z^2+2z^3+2z}\,\frac{dz}{z} \\
&= -\frac{i}{2}\int_{C_1(0)} \frac{z^4+1}{z^2(2z^2+5z+2)}dz \\
&= -\frac{i}{2}2\pi i \sum_j \operatorname{Res}\left(\frac{z^4+1}{z^2(2z^2+5z+2)},\, z_j\right) \\
&= \pi \sum_j \operatorname{Res}\left(\frac{z^4+1}{z^2(2z^2+5z+2)},\, z_j\right),
\end{aligned}$$

where the sum of the residues extends over all the poles of $\frac{z^4+1}{z^2(2z^2+5z+2)}$ inside the unit disk. We have a pole of order 2 at 0 and possible more poles at the roots of $2z^2+5z+2$. Let's compute the residue at 0.

$$\begin{aligned}\operatorname{Res}\left(\frac{z^4+1}{z^2(2z^2+5z+2)}, 0\right) &= \lim_{z\to 0}\frac{d}{dz}\frac{z^4+1}{(2z^2+5z+2)} \\ &= \frac{4z^3(2z^2+5z+2)-(z^4+1)(4z+5)}{(2z^2+5z+2)^2}\bigg|_{z=0} = -\frac{5}{4}.\end{aligned}$$

For the nonzero poles, solve

$$2z^2+5z+2=0.$$

The roots are $z=\frac{-5\pm 3}{4}$. Only $z_1=-\frac{1}{2}$ is inside $C_1(0)$. We compute the residue using Proposition 5.1.3(ii):

$$\begin{aligned}\operatorname{Res}\left(\frac{z^4+1}{z^2(2z^2+5z+2)}, z_1\right) &= \frac{z_1^4+1}{z_1^2}\frac{1}{\frac{d}{dz}(2z^2+5z+2)\big|_{z_1}} \\ &= \frac{(\frac{1}{2})^4+1}{\frac{1}{4}\left(4(-\frac{1}{2})+5\right)} = \frac{17}{12}.\end{aligned}$$

Hence

$$\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \pi\left(\frac{17}{12}-\frac{5}{4}\right)=\frac{\pi}{6}.$$

9. Let $z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$, $d\theta=\frac{-i\,dz}{z}$, $\cos\theta=\frac{z+1/z}{2}$, $\sin\theta=\frac{z-1/z}{2i}$. Then

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{7+2\cos\theta+3\sin\theta} &= -i\int_{C_1(0)}\frac{dz}{z(7+(z+1/z)+\frac{3}{2i}z^2-\frac{3}{2i}} \\ &= -i\int_{C_1(0)}\frac{dz}{(1-\frac{3}{2}i)z^2+7z+(1+\frac{3}{2}i)} \\ &= 2\pi\sum_j \operatorname{Res}\left(\frac{1}{(1-\frac{3}{2}i)z^2+7z+(1+\frac{3}{2}i)}, z_j\right),\end{aligned}$$

where the sum of the residues extends over all the poles of $\frac{1}{(1-\frac{3}{2}i)z^2+7z+(1+\frac{3}{2}i)}$ inside the unit disk. Solve

$$\left(1-\frac{3}{2}i\right)z^2+7z+\left(1+\frac{3}{2}i\right)=0.$$

You'll find

$$\begin{aligned}z &= \frac{-7\pm\sqrt{49-4(1-\frac{3}{2}i)(1+\frac{3}{2}i)}}{2(1-\frac{3}{2}i)} = \frac{-7\pm\sqrt{49-4(1+\frac{9}{4})}}{2(1-\frac{3}{2}i)} \\ &= \frac{-7\pm\sqrt{36}}{2-3i} = \frac{-7\pm 6}{2-3i} \\ &= \frac{-13}{2-3i} \text{ or } \frac{-1}{2-3i} \\ &= \frac{-13(2+3i)}{13} = -(2+3i) \text{ or } \frac{-(2+3i)}{13}.\end{aligned}$$

We have

$$|2+3i| = \sqrt{13} > 1 \text{ and } \left|\frac{-(2+3i)}{13}\right| = \frac{\sqrt{13}}{13} < 1.$$

So only $z_1 = \frac{-(2+3i)}{13}$ is inside $C_1(0)$. We compute the residue using Proposition 5.1.3(ii):

$$\begin{aligned}\operatorname{Res}(z_1) &= \frac{1}{2(1-\frac{3}{2}i)z_1+7} \\ &= \frac{1}{-2(1-\frac{3}{2}i)\frac{-(2+3i)}{13}+7} = \frac{1}{6}.\end{aligned}$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{7+2\cos\theta+3\sin\theta} = 2\pi\,\frac{1}{6} = \frac{\pi}{3}.$$

13.a. The solution will vary a little from what is in the text. Note the trick based on periodicity.

Step 1: Double angle formula

$$a+b\cos^2\theta = a+b\left(\frac{1+\cos 2\theta}{2}\right) = \frac{2a+b+b\cos 2\theta}{2},$$

so

$$\frac{1}{a+b\cos^2\theta} = \frac{2}{2a+b+b\cos 2\theta}.$$

Step 2. Change variables in the integral: $t = 2\theta$, $dt = 2d\theta$. Then

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos^2\theta} = \int_0^{4\pi} \frac{dt}{2a+b+b\cos t}.$$

The function $f(t) = \frac{1}{2a+b+b\cos t}$ is 2π-periodic. Hence its integral over intervals of length 2π are equal. So

$$I = \int_0^{2\pi} \frac{dt}{2a+b+b\cos t} + \int_{2\pi}^{4\pi} \frac{dt}{2a+b+b\cos t} = 2\int_0^{2\pi} \frac{dt}{2a+b+b\cos t}.$$

Step 3. Now use the method of Section 5.2 to evaluate the last integral. Let $z = e^{it}$, $dz = ie^{it}dt$, $dt = \frac{-i\,dz}{z}$, $\cos t = \frac{z+1/z}{2}$. Then

$$\begin{aligned}2\int_0^{2\pi} \frac{dt}{2a+b+b\cos t} &= -4i\int_{C_1(0)} \frac{dz}{bz^2+(4a+2b)z+b} \\ &= 8\pi\sum_j \operatorname{Res}\left(\frac{1}{bz^2+(4a+2b)z+b},\, z_j\right),\end{aligned}$$

where the sum of the residues extends over all the poles of $\frac{1}{bz^2+(4a+2b)z+b}$ inside the unit disk. Solve

$$bz^2+(4a+2b)z+b = 0,$$

and get

$$\begin{aligned} z &= \frac{-(4a+2b) \pm \sqrt{(4a+2b)^2 - 4b^2}}{2b} = \frac{-(2a+b) \pm 2\sqrt{a(a+b)}}{b} \\ &= z_1 = \frac{-(2a+b) + 2\sqrt{a(a+b)}}{b} \text{ or } z_2 = \frac{-(2a+b) - 2\sqrt{a(a+b)}}{b}. \end{aligned}$$

It is not hard to prove that $|z_1| < 1$ and $|z_2| > 1$. Indeed, for z_2, we have

$$|z_2| = \frac{2a+b}{b} + \frac{2\sqrt{a(a+b)}}{b} = 1 + \frac{2a}{b} + \frac{2\sqrt{a(a+b)}}{b} > 1$$

because a, b are > 0. Now the product of the roots of a quadratic equation $\alpha z^2 + \beta z + \gamma = 0$ ($\alpha \neq 0$) is always equal to $\frac{\gamma}{\alpha}$. Applying this in our case, we find that $z_1 \cdot z_2 = 1$, and since $|z_2| > 1$, we must have $|z_1| < 1$.

We compute the residue at z_1 using Proposition 1(ii), Sec. 5.1:

$$\begin{aligned} \operatorname{Res}\left(\frac{1}{bz^2 + (4a+2b)z + b},\ z_1 \right) &= \frac{1}{2bz_1 + (4a+2b)} \\ &= \frac{1}{-(2a+b)2 + 4\sqrt{a(a+b)} + (4a+2b)} \\ &= \frac{1}{4\sqrt{a(a+b)}}. \end{aligned}$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{a + \cos^2\theta} = 2\pi \frac{1}{4\sqrt{a(a+b)}} = \frac{2\pi}{\sqrt{a(a+b)}}.$$

Solutions to Exercises 5.3

1. Use the same contour as in Example 5.3.3. Several steps in the solution are very similar to those in Example 5.3.3; in particular, Steps 1, 2, and 4. Following the notation of Example 5.3.3, we have $I_{\gamma_R} = I_{[-R,\,R]} + I_{\sigma_R}$. Also, $\lim_{R\to\infty} I_{\sigma_R} = 0$, and

$$\lim_{R\to\infty} I_{[-R,\,R]} = I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

These assertions are proved in Example 5.3.3 and will not be repeated here. So all we need to do is evaluate I_{γ_R} for large values of R and then let $R \to \infty$. We have

$$I_{\gamma_R} = 2\pi i \sum_j \operatorname{Res}\left(\frac{1}{z^4+1},\; z_j\right),$$

where the sum ranges over all the residues of $\frac{1}{z^4+1}$ in the upper half-plane. The function $\frac{1}{z^4+1}$ have four (simple) poles. These are the roots of $z^4+1=0$ or $z^4=-1$. Using the result of Example 5.3.3, we find the roots to be

$$z_1 = \frac{1+i}{\sqrt{2}}, \qquad z_2 = \frac{-1+i}{\sqrt{2}}, \qquad z_3 = \frac{-1-i}{\sqrt{2}}, \qquad z_4 = \frac{1-i}{\sqrt{2}}.$$

In exponential form,

$$z_1 = e^{i\frac{\pi}{4}}, \qquad z_2 = e^{i\frac{3\pi}{4}}, \qquad z_3 = e^{i\frac{5\pi}{4}}, \qquad z_4 = e^{i\frac{7\pi}{4}}.$$

Only z_1 and z_2 are in the upper half-plane, and so inside γ_R for large $R > 0$. Using Proposition 5.1.2(ii) we write

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{z^4+1},\, z_1\right) &= \frac{1}{\frac{d}{dz}z^4+1\big|_{z=z_1}} = \frac{1}{4z_1^3} = \frac{1}{4}e^{-i\frac{3\pi}{4}};\\
\operatorname{Res}\left(\frac{1}{z^4+1},\, z_2\right) &= \frac{1}{\frac{d}{dz}z^4+1\big|_{z=z_2}} = \frac{1}{4z_2^3} = \frac{1}{4}e^{-i\frac{9\pi}{4}} = \frac{1}{4}e^{-i\frac{\pi}{4}}.
\end{aligned}$$

So

$$\begin{aligned}
I_{\gamma_R} &= 2\pi i\frac{1}{4}\left(e^{-i\frac{3\pi}{4}} + e^{-i\frac{\pi}{4}}\right)\\
&= \frac{\pi i}{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\\
&= \frac{\pi i}{2}(-i\sqrt{2}) = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Letting $R \to \infty$, we obtain $I = \frac{\pi}{\sqrt{2}}$.

5. The integral converges absolutely, as in Step 1 of Example 5.3.3. We will reason as in that example, and omit some of the details. Here $I_{\gamma_R} = I_{[-R,\,R]} + I_{\sigma_R}$; $\lim_{R\to\infty} I_{\sigma_R} = 0$, and

$$\lim_{R\to\infty} I_{[-R,\,R]} = I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}.$$

All we need to do is evaluate I_{γ_R} for large values of R and then let $R \to \infty$. The function $f(z) = \frac{1}{(z^2+1)^3}$ has poles of order 3 at $z_1 = i$ and $z_2 = -i$, but only z_1 is in the upper half-plane. You can evaluate the integral I_{γ_R} using the residue theorem; however, an equal good and perhaps faster way in this case is to use Cauchy's generalized integral formula (3.8.10). Write

$$I_{\gamma_R} = \int_{\gamma_R} \frac{1}{(z^2+1)^3} dz = \int_{\gamma_R} \frac{1}{[(z+i)(z-i)]^3} dz = \int_{\gamma_R} \frac{1}{(z+i)^3} \frac{dz}{(z-i)^3}.$$

Let $g(z) = \frac{1}{(z+i)^3}$. According to Cauchy's Integral Formula ,

$$I_{\gamma_R} = 2\pi i \frac{g''(i)}{2!}.$$

Compute:

$$g'(z) = -3(z+i)^{-4}, \qquad g''(z) = 12(z+i)^{-5},$$

so

$$g''(i) = 12(2i)^{-5} = \frac{12}{2^5}(-i).$$

Finally,

$$I_{\gamma_R} = \pi \frac{12}{2^5} = \frac{3\pi}{8}.$$

Letting $R \to \infty$, we obtain $I = \frac{3\pi}{8}$.

9. We use the same technique as in the solution of Exercise 5. This time, we have that $g(z) = \dfrac{1}{(z+i)^{n+1}}$.

Cauchy's Integral Formula gives $I_{\gamma_R} = 2\pi i \dfrac{g^{(n)}(i)}{n!}$.
Now

$$\begin{aligned} g^{(n)}(i) = \frac{d^n}{dz^n} \frac{1}{(z+i)^{n+1}}\bigg|_{z=i} &= \frac{(-1)^n (n+1)(n+2)\cdots 2n}{(z+i)^{(2n+1)}}\bigg|_{z=i} \\ &= \frac{(-1)^n (2n)!}{n!} \frac{1}{2^{2n+1} i^{2n+1}} \\ &= -\frac{i(2n)!}{2^{2n+1} n!}. \end{aligned}$$

Therefore, by Proposition 5.3.4

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} &= 2\pi i \operatorname{Res}\left(\frac{1}{(1+x^2)^{n+1}}, i \right) \\ &= 2\pi i \frac{g^{(n)}(i)}{n!} \\ &= -2\pi i \frac{i(2n)!}{2^{2n+1}(n!)^2} \\ &= \frac{(2n)!}{2^{2n}(n!)^2} \pi. \end{aligned}$$

13. Once again we follow Example 5.3.6 closely. First, we need to check that the integral is convergent, but it has been noted before (just above Example 5.3.6) that the integrals of the form

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{bx}+c}\,dx \quad \text{with } 0 < a < b \text{ and } c > 0$$

are, indeed convergent.
To calculate the poles of $f(x) = \frac{e^x}{3e^{2x}+1}$, we set

$$\begin{aligned} e^{bx}+1 = 0 \quad &\Leftrightarrow \quad e^{bx} = e^{i\pi} \\ &\Leftrightarrow \quad bx = i\pi + 2\pi i k, \quad \text{for } k \in \mathbb{Z} \end{aligned}$$

We use a contour as in Figure 5.18 and use the notation of Example 5.3.6. Setting $k = 0$, we have the pole $x = \frac{i\pi}{b}$. Therefore, we pick our rectangular contour to go from $y = 0$ to $y = \frac{2i\pi}{b}$. That way, we do not include all poles, but only the $x = \frac{i\pi}{b}$ one.
In this notation, we note that $\lim_{R\to\infty} |I_2| = 0$ and $\lim_{R\to\infty} |I_4| = 0$. Check Example 5.3.6 or the solution of Exercise 9 for details.

We do the calculation for I_3. There we have $u + vi = x + \frac{2i\pi}{b}$, from $x = R$ to $x = -R$. Therefore

$$I_3 = \int_R^{-R} \frac{e^{ax+\frac{2ai\pi}{b}}\,dx}{e^{bx+2i\pi}+1} = \int_R^{-R} \frac{e^{ax}e^{\frac{2a\pi i}{b}}\,dx}{e^{bx}+1} = -e^{\frac{2a\pi i}{b}} I_1$$

Then we apply the residue theorem to get

$$\operatorname{Res}\left(f(x), \frac{i\pi}{b}\right) = \lim_{x\to\frac{i\pi}{b}} \frac{\left(x-\frac{i\pi}{b}\right)e^{ax}}{e^{bx}+1} = -\frac{e^{\frac{ai\pi}{b}}}{b}$$

Finally, since

$$I_1 + I_2 + I_3 + I_4 = 2\pi i \operatorname{Res}\left(f(x), \frac{i\pi}{b}\right) = -\frac{2\pi i e^{\frac{ai\pi}{b}}}{b},$$

letting $R \to \infty$, we get that $I_1\left(1 - e^{\frac{2a\pi i}{b}}\right) = -\frac{2\pi i e^{\frac{ai\pi}{b}}}{b}$, meaning that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{ax}dx}{e^{bx}+1} &= \frac{-\frac{2\pi i e^{\frac{ai\pi}{b}}}{b}}{1-e^{\frac{2a\pi i}{b}}} \\ &= \frac{-2\pi i}{b} \frac{1}{e^{\frac{-a\pi i}{b}} - e^{\frac{a\pi i}{b}}} \\ &= \frac{-2\pi i}{b} \frac{1}{\left(\cos\left(\frac{a\pi}{b}\right) - i\sin\left(\frac{a\pi}{b}\right)\right) - \left(\cos\left(\frac{a\pi}{b}\right) + i\sin\left(\frac{a\pi}{b}\right)\right)} \\ &= \frac{\pi}{b\sin\left(\frac{a\pi}{b}\right)}. \end{aligned}$$

This implies that

$$\int_{-\infty}^{\infty} \frac{e^{ax}dx}{e^{bx}+1} = \frac{\pi}{b\sin\frac{a\pi}{b}}$$

when $0 < a < b$.

17. Let $x = e^t$, $dx = e^t\,dt$. Then

$$I = \int_0^\infty \frac{x^\alpha}{(x+1)^2}\,dx = \int_{-\infty}^\infty \frac{e^{\alpha t}}{(e^t+1)^2} e^t dt.$$

Use a rectangular contour γ_R as in Figure 5.18 with $\alpha = 1$. Then the vertical sides of the rectangle have length 2π and we have a pole of order 2 at πi which lies in the center of the rectangle. Refer to Example 5.3.6 for notation: $\lim_{R\to\infty} |I_2| = 0$, $\lim_{R\to\infty} |I_4| = 0$, and $\lim_{R\to\infty} I_1 = I$, the desired integral. For I_3, $z = x + i\pi$, x varies from R to $-R$, $dz = dx$, so

$$\begin{aligned} I_3 &= \int_R^{-R} \frac{e^{(\alpha+1)(x+2\pi i)}}{(e^{(x+2\pi i)}+1)^2}\,dx \\ &= -\int_{-R}^{R} \frac{e^{(\alpha+1)x} e^{2\pi i\alpha}}{(e^x+1)^2}\,dx \\ &= -e^{2\alpha\pi i} \int_{-R}^{R} \frac{e^{(\alpha+1)x}}{(e^x+1)^2}\,dx \\ &= -e^{2\alpha\pi i} I_1. \end{aligned}$$

We have

$$I_{\gamma_R} = \int_{\gamma_R} \frac{e^{(\alpha+1)z}}{(e^z+1)^2}\,dz = I_1 + I_2 + I_3 + I_4 = (1 - e^{2\alpha\pi i})I_1 + I_2 + I_4.$$

Letting $R \to \infty$ and using that $I_1 \to I$ and $I_2, I_4 \to 0$ as $R \to \infty$, we get

$$\lim_{R\to\infty} I_{\gamma_R} = (1 - e^{2\alpha\pi i})I.$$

To find the constant value of I_{γ_R} for large R, we calculate the residue at $z = \pi i$, since we have poles at $z = \pi i + 2k\pi i$ for $k \in \mathbb{Z}$, and only $z = \pi i$ is contained inside the contour. The value of this integral is

$$\begin{aligned} 2\pi i \operatorname{Res}\left(\frac{e^{(\alpha+1)z}}{(e^z+1)^2}, \pi i \right) &= 2\pi i \lim_{z\to\pi i} \frac{d}{dz} \frac{(z-\pi i)^2\, e^{(\alpha+1)z}}{(e^z+1)^2} \\ &= 2\pi i \lim_{z\to\pi i} \left[\frac{(\alpha+1)e^{(\alpha+1)z}}{(\frac{e^z+1}{z-\pi i})^2} - \frac{2\, e^{(\alpha+1)z}\, \frac{d}{dz}(\frac{e^z+1}{z-\pi i})}{(\frac{e^z+1}{z-\pi i})^3} \right] \\ &= 2\pi i \left[(\alpha+1)e^{i\pi(\alpha+1)} - 2 \cdot \frac{1}{2} e^{i\pi(\alpha+1)} \right] \\ &= 2\pi i\, \alpha e^{i\pi(\alpha+1)}, \end{aligned}$$

since

$$\lim_{z\to\pi i} \frac{d}{dz} \left(\frac{e^z+1}{z-\pi i} \right) = \frac{1}{2}.$$

It follows that

$$I = \frac{2\pi i\alpha e^{i\pi(\alpha+1)}}{(1 - e^{2\alpha\pi i})} = \frac{\pi\alpha}{\sin(\pi\alpha)}.$$

21. Let $ax = e^t$, $adx = e^t\,dt$. Then

$$I = \int_0^\infty \frac{\ln(ax)}{x^2+b^2}\,dx = \frac{1}{a}\int_{-\infty}^\infty \frac{t}{\frac{e^{2t}}{a^2}+b^2}e^t dt = a\int_{-\infty}^\infty \frac{xe^x}{e^{2x}+a^2b^2}dx.$$

Use a rectangular contour as in Figure 5.18, whose vertical sides have length π (i.e., $\alpha = 2$). Refer to Example 5.3.6 for notation: $\lim_{R\to\infty}|I_2| = 0$, $\lim_{R\to\infty}|I_4| = 0$, and $\lim_{R\to\infty} I_1 = I$, the desired integral. For I_3, $z = x + i\pi$, x varies from R to $-R$, $dz = dx$, so

$$\begin{aligned}
I_3 &= a\int_R^{-R} \frac{(x+i\pi)e^{x+i\pi}}{e^{2x+2\pi i}+a^2b^2}\,dx \\
&= -a\int_{-R}^{R} \frac{(x+i\pi)e^x(-1)}{e^{2x}+a^2b^2}\,dx \\
&= a\int_{-R}^{R} \frac{xe^x}{e^{2x}+a^2b^2}\,dx + \overbrace{2\pi i\int_{-R}^{R} \frac{e^x}{e^{2x}+a^2b^2}\,dx}^{=B_R i} = I_1 + B_R i,
\end{aligned}$$

where B_R is a real constant, because the integrand is real-valued. We have

$$I_{\gamma_R} = I_1 + I_2 + I_3 + I_4 = 2I_1 + I_2 + I_4 + iB_R.$$

Letting $R \to \infty$, we get

$$\lim_{R\to\infty} I_{\gamma_R} = 2I + iB,$$

where

$$B = \lim_{R\to\infty} B_R = \int_{-\infty}^\infty \frac{e^x}{e^{2x}+a^2b^2}\,dx.$$

At the same time

$$I_{\gamma_R} = 2\pi i \operatorname{Res}\left(\frac{2ze^z}{e^{2z}+a^2b^2}, z_0\right),$$

where z_0 is the root of $e^{2z} + a^2b^2 = 0$ that lies inside γ_R (there is only one root, as you will see):

$$\begin{aligned}
e^{2z} + a^2b^2 = 0 \quad &\Rightarrow \quad e^{2z} = e^{2\ln(ab)+i\pi} \\
&\Rightarrow \quad 2z = 2\ln(ab) + i\pi + 2k\pi i \\
&\Rightarrow \quad z = \ln(ab) + i\frac{\pi}{2}(2k+1).
\end{aligned}$$

Only $z = 2\ln 2 + i\frac{\pi}{2}$ is inside the contour. So

$$\begin{aligned}
I_{\gamma_R} &= 2\pi i \operatorname{Res}\left(\frac{aze^z}{e^{2z}+a^2b^2}, \ln(ab) + i\frac{\pi}{2}\right) \\
&= 2\pi i \frac{a(\ln(ab)+i\frac{\pi}{2})e^{\ln(ab)+i\frac{\pi}{2}}}{2e^{2(\ln(ab)+i\frac{\pi}{2})}} \\
&= \frac{\pi}{b}\left(\ln(ab) + i\frac{\pi}{2}\right)
\end{aligned}$$

Thus

$$2I + 2\pi i B = \frac{\pi}{b}\ln(ab) + i\frac{\pi^2}{2b}.$$

Taking real and imaginary parts, we find

$$I = \frac{\pi \ln(ab)}{2b} \qquad \text{and} \qquad B = \frac{\pi}{4b}.$$

This gives the value of the desired integral I and also of the integral

$$\frac{\pi}{4b} = B = \int_{-\infty}^{\infty} \frac{e^x}{e^{2x} + a^2b^2}\, dx.$$

25. The integral converges by the comparison test for convergence ($\int_1^\infty \frac{1}{x^3} dx$ converges). The integrand has poles at $x = -1, e^{\pi i}$ and $e^{-\pi i}$. We calculate that

$$\text{Res}\left(\frac{1}{x^3+1}, e^{\pi i}\right) = \lim_{x \to e^{\pi i}} \frac{x - e^{\pi i}}{x^3+1} = \frac{1}{3e^{\frac{2\pi i}{3}}}$$

Using the given, we get that

$$\int_{\gamma_2} \frac{1}{x^3+1} dx = \int_0^{\frac{2\pi}{3}} \frac{Re^t}{(Re^t)^3+1} dt = \frac{1}{R^2}\int_0^{\frac{2\pi}{3}} \frac{e^t}{e^{2t}+R^{-3}} dt \to 0$$

as $R \to \infty$, because

$$\left|\int_0^{\frac{2\pi}{3}} \frac{e^t}{e^{2t}+R^{-3}} dt\right| \le \left|\int_0^{\frac{2\pi}{3}} \frac{1}{e^t} dt\right| < \infty.$$

Therefore

$$\lim_{R\to\infty} I_2 = \lim_{R\to\infty} \int_{\gamma_2} \frac{1}{x^3+1} = 0.$$

Moreover, in γ_3, we have $x = te^{\frac{2\pi i}{3}}$, for t from R to 0 and therefore $dx = e^{\frac{2\pi i}{3}} dt$.

Thus

$$\int_{\gamma_3} \frac{1}{x^3+1} dx = e^{\frac{2\pi i}{3}} \int_R^0 \frac{1}{\left(te^{\frac{2\pi i}{3}}\right)^3+1} dt = -e^{\frac{2\pi i}{3}} \int_{\gamma_1} \frac{1}{x^3+1} dx.$$

The Cauchy residue theorem gives us

$$I_1 + I_2 + I_3 = 2\pi i \ \text{Res}\left(\frac{1}{x^3+1}, e^{\pi i}\right) = \frac{2\pi i}{3e^{\frac{2\pi i}{3}}}$$

and therefore, since $I_3 = -e^{\frac{2\pi i}{3}} I_1$ and $I_2 \to 0$,

$$\begin{aligned}\int_0^\infty \frac{1}{x^3+1} dx &= \lim_{R\to\infty} I_1 \\ &= \frac{1}{1+e^{\frac{2\pi i}{3}}} \frac{2\pi i}{3e^{\frac{2\pi i}{3}}} \\ &= \frac{2\pi}{3\sqrt{3}}.\end{aligned}$$

25. (a) Since $0 < a < 1$, the integral converges, by the comparison test. Indeed, since $a > 0$, by the comparison test, we have that

$$\Big|\int_1^\infty \frac{x^{a-1}}{x+1}dx\Big| \leq \Big|\int_1^\infty \frac{1}{x^{2-a}}dx\Big| < \infty.$$

Moreover, since $a < 1$, Let $x = e^t$. Then $dx = e^t dt$ and

$$\Big|\int_0^1 \frac{x^{a-1}}{x+1}dx\Big| \leq \Big|\int_0^1 \frac{1}{x^{1-a}}dx\Big| < \infty.$$

Remember that if $c > 0$, $\int_c^\infty \frac{1}{x^b}dx$ converges if and only if $b > 1$, where as $\int_0^c \frac{1}{x^b}dx$ converges if and only if $b < 1$.

Let $x = e^t$. Then $dx = e^t\, dt$ and

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx = \int_{-\infty}^\infty \frac{e^{(a-1)t}}{e^t+1}\, e^t dt = \int_{-\infty}^\infty \frac{e^{at}}{e^t+1}dt.$$

The result then follows from Exercise 13.
(b) The definition of Gamma is

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,$$

for Re $z > 0$. Therefore,

$$\Gamma(a)\Gamma(a-1) = \int_0^\infty e^{-t}t^{a-1}dt \int_0^\infty e^{-s}s^{(1-a)-1}ds = \int_0^\infty \int_0^\infty e^{-(t+s)}t^{a-1}s^{-a}dsdt.$$

(c) Let $x = s + t$ and $y = \frac{t}{s}$. Then $t = \frac{xy}{y+1}$, $s = \frac{x}{y+1}$, and

$$\frac{\partial(t,s)}{\partial(x,y)} = \frac{\partial t}{\partial x}\frac{\partial s}{\partial y} - \frac{\partial s}{\partial x}\frac{\partial t}{\partial y}$$

We calculate that:

$$\begin{aligned}
\frac{\partial t}{\partial x} &= \frac{y}{y+1}\\
\frac{\partial t}{\partial y} &= \frac{x}{(y+1)^2}\\
\frac{\partial s}{\partial x} &= \frac{1}{y+1}\\
\frac{\partial y}{\partial s} &= -\frac{x}{(y+1)^2}
\end{aligned}$$

Therefore

$$\Big|\frac{\partial(x,y)}{\partial(t,s)}\Big| = \Big|\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial t}\Big| = \Big|\frac{y}{y+1}\left(-\frac{x}{(y+1)^2}\right) - \frac{x}{(y+1)^2}\frac{1}{y+1}\Big| = \frac{x}{(y+1)^2}$$

since $x, y > 0$ in the integral. We use the change of variables formula to get

$$
\begin{aligned}
\int_0^\infty \int_0^\infty e^{-(t+s)} t^{a-1} s^{-a} ds dt &= \int_0^\infty \int_0^\infty e^{-(t+s)} \frac{t}{s}^a t^{-1} ds dt \\
&= \int_0^\infty \int_0^\infty e^{-x} y^a \left(\frac{xy}{y+1}\right)^{-1} \left|\frac{\partial(x,y)}{\partial(t,s)}\right| dy dx \\
&= \int_0^\infty \int_0^\infty e^{-x} y^a \frac{y+1}{xy} \frac{x}{(y+1)^2} dy dx \\
&= \int_0^\infty \int_0^\infty e^{-x} y^{a-1} \frac{1}{(y+1)} dy dx \\
&= \int_0^\infty e^{-x} \left(\int_0^\infty y^{a-1} \frac{1}{(y+1)} dy\right) dx \\
&= \int_0^\infty e^{-x} \left(\frac{\pi}{\sin \pi a}\right) dx \\
&= \frac{\pi}{\sin \pi a} \int_0^\infty e^{-x} dx \\
&= \frac{\pi}{\sin \pi a}.
\end{aligned}
$$

(d) We know that Γ is holomorphic for $\operatorname{Re} z > 0$. Therefore, if $0 < z < 1$, $\Gamma(z)\Gamma(1-z)$ is also holomorphic.
On the other hand, the function $f(z) = \frac{\pi}{\sin \pi z}$ is also holomorphic for $0 < z < 1$, since $\sin \pi a \neq 0$ there. Since the two functions are equal on the real line, we conclude, using the identity principle, that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

for all $0 < z < 1$.

(e) Exercise 25 in Section 4.2 gives an outline of the proof for the formula

$$\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = 2\int_0^{\frac{\pi}{2}} (\cos^{2z_1-1}\theta)(\sin^{2z_2-1}\theta) d\theta$$

By picking $z_1 = \frac{3}{4}$ and $z_2 = \frac{1}{4}$, we get that

$$\begin{aligned}
2\int_0^{\frac{\pi}{2}} \sqrt{\cot\,\theta}\;d\theta &= 2\int_0^{\frac{\pi}{2}} (\cos^{2\frac{3}{4}}\,-1\,\theta)(\sin^{2\frac{1}{4}-1}\;\theta)\;d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (\cos^{2z_1-1}\;\theta)(\sin^{2z_2-1}\;\theta)d\theta \\
&= \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} \\
&= \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}+\frac{1}{4})} \\
&= \frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{\Gamma(1)} \\
&= \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \\
&= \frac{\pi}{\sin\;\pi\frac{1}{4}} \\
&= \frac{2\pi}{\sqrt{2}}.
\end{aligned}$$

Therefore

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\;\theta}\;d\theta = \frac{\pi}{\sqrt{2}}.$$

Solutions to Exercises 5.4

1. By (5.4.2), we have that

$$\int_{-\infty}^{\infty} \frac{\cos 4x}{x^2+1}\,dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{4ix}}{x^2+1}\,dx\right).$$

Consider the contour integral

$$I_{\gamma_R} = \int_{\gamma_R} \frac{e^{4iz}}{z^2+1}\,dz = \int_{\sigma_R} \frac{e^{4iz}}{z^2+1}\,dz + \int_{-R}^{R} \frac{e^{4ix}}{x^2+1}\,dx = I_{\sigma_R} + I_R,$$

where γ_R and σ_R are as in Figure 5.21. For $R > 1$, the integrand has one simple pole at $z = i$. By Proposition 5.1.3(ii), we have that

$$\operatorname{Res}\left(\frac{e^{4iz}}{z^2+1}, i\right) = \frac{e^{-4}}{2i}.$$

Thus, by the residue theorem, for all $R > 1$, we have

$$I_{\gamma_R} = I_{\sigma_R} + I_R = 2\pi i \frac{e^{-4}}{2i} = \pi e^{-4}.$$

We claim that $I_{\sigma_R} \to 0$ as $R \to \infty$. On the contour σ_R, we have that $z = R(\cos\theta + i\sin\theta)$, $0 \le \theta \le \pi$. Thus, $\sin\theta \ge 0$ on the contour. We then have that

$$\left|e^{4iz}\right| = \left|e^{4iR(\cos\theta + i\sin\theta)}\right| = e^{-4R\sin\theta} \le 1.$$

We can then estimate the integral I_{σ_R} by noting that

$$\left|\frac{e^{4iz}}{z^2+1}\right| \le \frac{1}{|z^2+1|} \le \frac{1}{|z|^2-1} = \frac{1}{R^2-1},$$

so by the ML-inequality,

$$|I_{\sigma_R}| = \left|\int_{\sigma_R} \frac{e^{4iz}}{z^2+1}\,dz\right| \le \frac{\pi R}{R^2-1} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus, as $R \to \infty$, $I_R \to I_{\gamma_R} = \pi e^{-4}$. We then have that

$$\int_{-\infty}^{\infty} \frac{\cos 4x}{x^2+1}\,dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{4ix}}{x^2+1}\,dx\right) = \operatorname{Re}\left(\pi e^{-4}\right) = \pi e^{-4}.$$

5. The degree of the denominator is 2 more than the degree of the numerator; so we can use the contour in Figure 5.21 and proceed as in Example 5.4.1.
Step 1: The integral is absolutely convergent.

$$\left|\frac{x^2\cos 2x}{(x^2+1)^2}\right| \le \frac{x^2}{(x^2+1)^2} \le \frac{1}{x^2+1},$$

because
$$\frac{x^2}{(x^2+1)^2} = \frac{(x^2+1)-1}{(x^2+1)^2} = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2} \leq \frac{1}{x^2+1}.$$
Since $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx < \infty$ (you can actually compute the integral $= \pi$), we conclude that our integral is absolutely convergent.

Step 2:
$$\int_{-\infty}^{\infty} \frac{x^2\cos 2x}{(x^2+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{x^2\cos 2x}{(x^2+1)^2}\,dx + i\int_{-\infty}^{\infty} \frac{x^2\sin 2x}{(x^2+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{x^2 e^{2ix}}{(x^2+1)^2}\,dx$$
because
$$\int_{-\infty}^{\infty} \frac{x^2\sin 2x}{(x^2+1)^2}\,dx = 0,$$
being the integral of an odd function over a symmetric interval.

Step 3: Let γ_R and σ_R be as in Figure 5.21. We will show that
$$\int_{\sigma_R} \frac{z^2 e^{2iz}}{(z^2+1)^2}\,dz \to 0 \text{ as } R \to \infty.$$
$$|I_{\sigma_R}| = \left|\int_{\sigma_R} \frac{z^2 e^{2iz}}{(z^2+1)^2}\,dz\right| \leq l(\sigma_R) \max_{z \text{ on } \sigma_R} \left|\frac{z^2 e^{2iz}}{(z^2+1)^2}\right| = \pi R \cdot M.$$
For z on σ_R we have
$$\left|e^{2iz}\right| \leq e^{-2R\sin\theta} \leq 1.$$
So
$$\begin{aligned}
\left|\frac{z^2 e^{2iz}}{(z^2+1)^2}\right| &\leq \left|\frac{z^2}{(z^2+1)^2}\right| \leq \left|\frac{z^2+1-1}{(z^2+1)^2}\right| \\
&\leq \left|\frac{1}{z^2+1}\right| + \left|\frac{1}{(z^2+1)^2}\right| \\
&\leq \frac{1}{R^2-1} + \frac{1}{(R^2-1)^2}.
\end{aligned}$$
So
$$|I_{\sigma_R}| = \frac{\pi R}{R^2-1} + \frac{\pi R}{(R^2-1)^2},$$
and this goes to zero as $R \to \infty$.

Step 4: We have
$$\int_{\gamma_R} \frac{z^2 e^{2iz}}{(z^2+1)^2}\,dz = 2\pi\,\mathrm{Res}\left(\frac{z^2 e^{2iz}}{(z^2+1)^2},\ i\right),$$
because we have only one pole of order 2 at i in the upper half-plane.
$$\begin{aligned}
\mathrm{Res}\left(\frac{z^2 e^{2iz}}{(z^2+1)^2},\ i\right) &= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z^2 e^{2iz}}{(z^2+1)^2}\right] \\
&= \frac{d}{dz}\left[\frac{z^2 e^{2iz}}{(z+i)^2}\right]\Bigg|_{z=i} \\
&= i\frac{e^{-2}}{4},
\end{aligned}$$

after many (hard-to-type-but-easy-to-compute) steps that we omit. So $I_{\gamma_R} = 2\pi i(i\frac{e^{-2}}{4}) = -\pi\frac{e^{-2}}{2}$. Letting $R \to \infty$ and using the fact that $I_{\gamma_R} \to I$, the desired integral, we find $I = -\pi\frac{e^{-2}}{2}$.

9. In the integral the degree of the denominator is only one more than the degree of the numerator. So the integral converges in the principal value sense. Let us check if the denominator has roots on the real axis:

$$x^2 + x + 9 = 0 \Rightarrow x = -\frac{1}{2} \pm i\frac{\sqrt{35}}{2}.$$

We have no roots on the real axis, so we will proceed as in Example 5.4.5, and use Jordan's Lemma. Refer to Example 5.4.5 for further details of the solution. Consider

$$\int_{\gamma_R} \frac{z}{z^2+z+9} e^{i\pi z}\, dz = \int_{\gamma_R} f(z)e^{i\pi z}\, dz,$$

where γ_R is as in Figure 5.24. By Corollary 5.4.4,

$$\left| \int_{\sigma_R} f(z)e^{i\pi z}\, dz \right| \to 0, \text{ as } R \to \infty.$$

Apply the residue theorem:

$$\int_{\gamma_R} f(z)e^{i\pi z}\, dz = 2\pi i \operatorname{Res}\left(f(z)e^{i\pi z}, z_1\right),$$

where $z_1 = -\frac{1}{2} + i\frac{\sqrt{35}}{2}$ is the only (simple) pole of $f(z)e^{i\pi z}$ in the upper half-plane. By Proposition 5.1.3(ii) we have

$$\begin{aligned}
\operatorname{Res}\left(f(z)e^{i\pi z}, z_1\right) &= \left.\frac{ze^{i\pi z}}{\frac{d}{dz}(z^2+z+9)}\right|_{z=z_1} \\
&= \frac{z_1 e^{i\pi z_1}}{2z_1+1} = \frac{-(\frac{1}{2} + \frac{\sqrt{35}}{2} i)}{\sqrt{35}} e^{-\pi\frac{\sqrt{35}}{2}} \\
&= -\frac{e^{-\pi\frac{\sqrt{35}}{2}}}{2\sqrt{35}} - i\frac{e^{-\pi\frac{\sqrt{35}}{2}}}{2}.
\end{aligned}$$

So

$$\begin{aligned}
\int_{\gamma_R} f(z)e^{i\pi z}\, dz &= 2\pi i \left(-\frac{e^{-\pi\frac{\sqrt{35}}{2}}}{2\sqrt{35}} - i\frac{e^{-\pi\frac{\sqrt{35}}{2}}}{2} \right) \\
&= \pi e^{-\pi\frac{\sqrt{35}}{2}} + i\pi\frac{e^{-\pi\frac{\sqrt{35}}{2}}}{\sqrt{35}}
\end{aligned}$$

But

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)e^{i\pi z}\, dz = \int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+x+9}\, dx + i\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+x+9}\, dx.$$

Taking real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+x+9}\,dx = \pi e^{-\pi\frac{\sqrt{35}}{2}} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+x+9}\,dx = \pi\frac{e^{-\pi\frac{\sqrt{35}}{2}}}{\sqrt{35}}.$$

13. Use an indented contour $\gamma_{r,R}$ as in Fig. 5.31, with an indentation around 0, and consider the integral

$$I_{r,R} = \int_{\gamma_{r,R}} \frac{1-e^{iz}}{z^2}\,dz = \int_{\gamma_{r,R}} g(z)\,dz,$$

where $g(z) = \frac{1-e^{iz}}{z^2}$. Note that $g(z)$ has a simple pole at 0. To see this, consider its Laurent series expansion around 0:

$$\begin{aligned} g(z) &= \frac{1}{z^2}\left(1-\left(1+(iz)+(iz)^2/2!+(iz)^3/3!+\cdots\right)\right) \\ &= -\frac{i}{z}+\frac{1}{2}+i\frac{z}{3!}-\cdots \end{aligned}$$

Moreover, $\operatorname{Res}(g(z),\ 0) = -i$. By Corollary 5.4.8,

$$\lim_{r\to 0^+}\int_{\sigma_r} g(z) = i\pi(-i) = \pi.$$

(Keep in mind that σ_r has a positive orientation, so it is traversed in the opposit direction on $\gamma_{r,R}$. See Figure 5.31.) On the outer semi-circle, we have

$$\left|\int_{\sigma_R} g(z)\,dz\right| \le \frac{\pi R}{R^2}\max_{z \text{ on } \sigma_R}|1-e^{iz}| = \frac{\pi}{R}\max_{z \text{ on } \sigma_R}|1-e^{iz}|.$$

Using an estimate as in Example 5.4.1, Step 3, we find that for z on σ_R,

$$\left|e^{iz}\right| \le e^{-R\sin\theta} \le 1.$$

Hence $\max_{z \text{ on } \sigma_R}|1-e^{iz}| \le 1+1 = 2$. So $|I_{\sigma_R}| \le \frac{2\pi}{R} \to 0$, as $R\to\infty$. Now $g(z)$ is analytic inside and on the simple path $\gamma_{r,R}$. So by Cauchy's theorem,

$$\int_{\gamma_{r,R}} g(z)\,dz = 0.$$

So

$$0 = \int_{\gamma_{r,R}} g(z)\,dz = \overbrace{\int_{\sigma_R} g(z)\,dz}^{\to 0} - \overbrace{\int_{\sigma_r} g(z)\,dz}^{\to\pi} + \overbrace{\int_{[-R,-r]} g(z)\,dz + \int_{[r,\ R]} g(z)\,dz}^{\to I}.$$

As $r\to 0^+$ and $R\to\infty$, we obtain

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{1-\cos x}{x^2}\,dx = \operatorname{Re}(I) = \operatorname{Re}(\pi) = \pi.$$

17. Use an indented contour as in Figure 5.31 with an indentation around 0. We have

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+1)}\,dx = \operatorname{Im}\left(\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx\right) = \operatorname{Im}(I).$$

$$I_{r,R} = \int_{\gamma_{r,R}}\frac{e^{iz}}{z(z^2+1)}\,dz = \int_{\gamma_{r,R}} g(z)\,dz,$$

where $g(z) = \frac{e^{iz}}{z(z^2+1)}$. Note that $g(z)$ has a simple pole at 0, and $\operatorname{Res}(g(z),\ 0) = 1$. By Corollary 2,

$$\lim_{r\to 0^+}\int_{\sigma_r} g(z) = i\pi(1) = i\pi.$$

(As in Exercise 13, σ_r has a positive orientation, so it is traversed in the opposit direction on $\gamma_{r,R}$. See Figure 5.31.) As in Example 1, $|I_{\sigma_R}| \to 0$, as $R\to\infty$. Now $g(z)$ has a simple pole at i inside $\gamma_{r,R}$. So by the residue theorem,

$$\int_{\gamma_{r,R}} g(z)\,dz = 2\pi\operatorname{Res}(g(z),\ i) = 2\pi i\,\frac{e^{i(i)}}{i(2i)} = -\pi e^{-1}.$$

So

$$-\pi e^{-1} = \int_{\gamma_{r,R}} g(z)\,dz = \overbrace{\int_{\sigma_R} g(z)\,dz}^{\to 0} - \overbrace{\int_{\sigma_r} g(z)\,dz}^{\to i\pi} + \overbrace{\int_{[-R,\,-r]} g(z)\,dz + \int_{[r,\ R]} g(z)\,dz}^{\to I}.$$

As $r\to 0^+$ and $R\to\infty$, we obtain

$$-\pi e^{-1} = I - i\pi.$$

Solving for I and taking imaginary parts, we find

$$I = i(\pi - \pi e^{-1}) \qquad \operatorname{Im}(I) = \pi - \pi e^{-1},$$

which is the value of the desired integral.

21. (a) We have that $\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi$, so since $\frac{\sin x}{x}$ is even, $\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}$. Trivially, for $a = 0$, we have $\int_0^{\infty}\frac{\sin ax}{x}\,dx = 0$. For $a > 0$, let $u = ax$ to get

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin ax}{x}\,dx = \frac{2}{\pi}\int_0^{\infty}\frac{\sin u}{\frac{u}{a}}\frac{du}{a} = \frac{2}{\pi}\int_0^{\infty}\frac{\sin u}{u}\,du = 1.$$

For $a < 0$, letting $u = ax$ changes the limits of integration, yielding

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin ax}{x}\,dx = \frac{2}{\pi}\int_0^{-\infty}\frac{\sin u}{\frac{u}{a}}\frac{du}{a} = -\frac{2}{\pi}\int_0^{\infty}\frac{\sin u}{u}\,du = -1.$$

(b) We have that

$$(\sin ax)(\cos bx) = \frac{1}{2}\Big(\sin\big((a+b)x\big) + \sin\big((a-b)x\big)\Big).$$

Thus, the integral becomes

$$\begin{aligned}\int_0^\infty \frac{(\sin ax)(\cos bx)}{x}\,dx &= \int_0^\infty \frac{\sin\big((a+b)x\big)+\sin\big((a-b)x\big)}{2x}\,dx\\ &= \int_0^\infty \frac{\sin\big((a+b)x\big)}{2x}\,dx + \int_0^\infty \frac{\sin\big((a-b)x\big)}{2x}\,dx\\ &= \frac{\pi}{4}\big(\operatorname{sgn}(a+b)+\operatorname{sgn}(a-b)\big) = \frac{\pi}{4}+\frac{\pi}{4}\operatorname{sgn}(a-b),\end{aligned}$$

from which the claimed formula follows.

25. (a) For $w > 0$, consider

$$J = \int_C \frac{e^{iwz}}{e^{2\pi z}-1}\,dz,$$

where C is the indented contour in Figure 5.32. The integrand is analytic inside and on C. So

$$(1) \qquad 0 = J = \int_C \frac{e^{iwz}}{e^{2\pi z}-1}\,dz = I_1+I_2+I_3+I_4+I_6+I_6,$$

where I_j is the integral over the jth component of C, starting with the line segement $[\epsilon,\ R]$ and moving around C counterclockwise. As $\epsilon \to 0^+$ and $R \to \infty$, $I_1 \to I$, the desired integral.

For I_3, $z = x + i$, where x varies from R to ϵ:

$$I_3 = \int_R^\epsilon \frac{e^{iw(x+i)}}{e^{2\pi(x+i)}-1}dx = -e^{-w}\int_\epsilon^R \frac{e^{iwx}}{e^{2\pi x}-1}dx.$$

As $\epsilon \to 0^+$ and $R \to \infty$, $I_3 \to -e^{-w}I$.

For I_2, $z = R + iy$, where y varies from 0 to 1:

$$|I_3| \le 1\cdot \max_{\substack{z=R+iy\\ 0\le y\le 1}} \left|\frac{e^{iwz}}{e^{2\pi z}-1}\right|;$$

$$\begin{aligned}\left|\frac{e^{iwz}}{e^{2\pi z}-1}\right| &= \left|\frac{e^{iw(R+iy)}}{e^{2\pi(R+iy)}-1}\right|\\ &= \left|\frac{e^{iwR}}{e^{2\pi R}-1}\right| \to 0,\ \text{ as } R\to\infty.\end{aligned}$$

So $I_2 \to 0$, as $R \to \infty$.

For I_5, $z = iy$, where y varies from $1-\epsilon$ to ϵ:

$$I_5 = \int_{1-\epsilon}^{\epsilon} \frac{e^{iw(iy)}}{e^{2\pi(iy)}-1} i\,dy = -i\int_\epsilon^{1-\epsilon} \frac{e^{-wy}}{e^{2\pi iy}-1}\,dy.$$

For I_4, the integral over the quarter circe from $(\epsilon, \epsilon + i)$ to $(0,\ i - i\epsilon)$, we apply Corollary 5.4.8, to compute the limit

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\gamma_4}\frac{e^{iwz}}{e^{2\pi z}-1}dz &= -i\frac{\pi}{2}\operatorname{Res}\left(\frac{e^{iwz}}{e^{2\pi z}-1},\ i\right)\\ &= -i\frac{\pi}{2}\frac{e^{iw(i)}}{2\pi e^{2\pi(i)}} = -i\frac{e^{-w}}{4}.\end{aligned}$$

For I_6, the integral over the quarter circe from $(0,\ i\epsilon)$ to $(\epsilon,\ 0)$, we apply Corollary 5.4.8, to compute the limit

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\gamma_6}\frac{e^{iwz}}{e^{2\pi z}-1}dz &= -\frac{\pi}{2}\operatorname{Res}\left(\frac{e^{iwz}}{e^{2\pi z}-1},\ 0\right)\\ &= -\frac{\pi}{2}i\frac{e^{iw(0)}}{2\pi e^{2\pi(0)}} = -\frac{i}{4}.\end{aligned}$$

Plug these findings in (1) and take the limit as $R \to \infty$ then as $\epsilon \to 0$, and get

$$\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - e^{-w}\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - i\int_\epsilon^{1-\epsilon}\frac{e^{-wy}}{e^{2\pi iy}-1}\,dy\right] - i\frac{e^{-w}}{4} - \frac{i}{4} = 0$$

or

$$\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - e^{-w}\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - i\int_\epsilon^{1-\epsilon}\frac{e^{-wy}}{e^{2\pi iy}-1}\,dy\right] = i(\frac{e^{-w}}{4} + \frac{1}{4}).$$

(Note: The limits of each individual improper integral does not exist. But the limit of the sum, as shown above does exist. So, we must work wth the limit of the three terms together.) Take imaginary parts on both sides:

$$(2)\quad \lim_{\epsilon\to 0}\operatorname{Im}\left[\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - e^{-w}\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - i\int_\epsilon^{1-\epsilon}\frac{e^{-wy}}{e^{2\pi iy}-1}\,dy\right] = \frac{e^{-w}+1}{4}.$$

Now

$$\begin{aligned}&\operatorname{Im}\left[\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - e^{-w}\int_\epsilon^\infty \frac{e^{iwx}}{e^{2\pi x}-1}dx - i\int_\epsilon^{1-\epsilon}\frac{e^{-wy}}{e^{2\pi iy}-1}\,dy\right]\\ (3)\qquad &= \int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}(1-e^{-w})\,dx + \int_\epsilon^{1-\epsilon}\operatorname{Im}\left(\frac{(-i)e^{-wy}}{e^{2\pi iy}-1}\right)\,dy;\end{aligned}$$

$$\begin{aligned}\operatorname{Im}\left(\frac{(-i)e^{-wy}}{e^{2\pi iy}-1}\right) &= -\operatorname{Re}\left(\frac{(-i)e^{-wy}}{e^{2\pi iy}-1}\right)\\ &= -\operatorname{Re}\left(\frac{e^{-wy}}{e^{2\pi iy}-1}\right)\\ &= -e^{-wy}\operatorname{Re}\left(\frac{1}{e^{2\pi iy}-1}\right) = -e^{-wy}\operatorname{Re}\left(\frac{e^{-i\pi y}}{e^{i\pi y}-e^{-i\pi y}}\right)\\ &= -e^{-wy}\operatorname{Re}\left(\frac{\cos\pi y - i\sin\pi y}{2i\sin\pi y}\right) = \frac{e^{-wy}}{2}.\end{aligned}$$

Plugging this in (3) and using (2), we get

$$\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}(1-e^{-w})\,dx + \int_\epsilon^{1-\epsilon}\frac{e^{-wy}}{2}\,dy\right] = \frac{e^{-w}+1}{4}.$$

Evaluate the second integral and get

$$\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}(1-e^{-w})\,dx + \frac{e^{-w(1-\epsilon)}-e^{-w\epsilon}}{2(-w)}\right] = \frac{e^{-w}+1}{4}.$$

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}(1-e^{-w})\,dx + \lim_{\epsilon\to 0}\frac{e^{-w(1-\epsilon)}-e^{-w\epsilon}}{2(-w)} = \frac{e^{-w}+1}{4}.$$

$$(1-e^{-w})\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}\,dx + \frac{1-e^{-w}}{2w} = \frac{e^{-w}+1}{4}.$$

So

$$\begin{aligned}
(1-e^{-w})\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}\,dx &= \frac{e^{-w}+1}{4} - \frac{1-e^{-w}}{2w};\\
\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\sin wx}{e^{2\pi x}-1}\,dx &= \frac{1}{1-e^{-w}}\left[\frac{e^{-w}+1}{4} - \frac{1-e^{-w}}{2w}\right]\\
\int_0^\infty \frac{\sin wx}{e^{2\pi x}-1}\,dx &= \frac{-1}{2w} + \frac{1}{4}\frac{e^{w}+1}{e^{w}-1}.
\end{aligned}$$

(b) Setting $B_0 = 1$ we have

$$\begin{aligned}
z\coth z &= \sum_{n=0}^\infty 2^{2n}\frac{B_{2n}}{(2n)!}z^{2n}, \quad |z|<\pi\\
\frac{z}{2}\coth\frac{z}{2} &= \sum_{n=0}^\infty 2^{2n}\frac{B_{2n}}{(2n)!}\left(\frac{z}{2}\right)^{2n}, \quad \left|\frac{z}{2}\right|<\pi\\
\frac{z}{2}\frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}} &= \sum_{n=0}^\infty \frac{B_{2n}}{(2n)!}z^{2n}, \quad |z|<2\pi\\
\frac{1}{2}\frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}} &= \frac{B_0}{z} + \sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}z^{2n-1}, \quad |z|<2\pi\\
\frac{1}{2}\frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}} - \frac{B_0}{z} &= \sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}z^{2n-1}, \quad |z|<2\pi\\
\frac{1}{4}\frac{e^{z}+1}{e^{z}-1} - \frac{1}{2z} &= \frac{1}{2}\sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}z^{2n-1}, \quad |z|<2\pi
\end{aligned}$$

(c) Replace $\sin wx$ in the integral in (a) by its Taylor series

$$\sin wx = \sum_{k=1}^\infty (-1)^{k-1}\frac{w^{2k-1}x^{2k-1}}{(2k-1)!},$$

using part (b) and interchanging order of integration we get

$$\begin{aligned}
\int_0^\infty \frac{1}{e^{2\pi x}-1}\sum_{n=0}^\infty (-1)^n \frac{(wx)^{2n+1}}{(2n+1)!}\,dx &= \frac{1}{2}\sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}w^{2n-1} \\
\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}w^{2n+1}\int_0^\infty \frac{x^{2n+1}}{e^{2\pi x}-1}\,dx &= \frac{1}{2}\sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}w^{2n-1} \\
\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)!}w^{2n-1}\int_0^\infty \frac{x^{2n-1}}{e^{2\pi x}-1}\,dx &= \frac{1}{2}\sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}w^{2n-1}.
\end{aligned}$$

Comparing the coefficients of w, we obtain

$$\int_0^\infty \frac{x^{2n-1}}{e^{2\pi x}-1}\,dx = \frac{(-1)^{n-1}}{4n}B_{2n} \quad (n = 1,\ 2,\ \ldots).$$

Solutions to Exercises 5.5

1. As in Example 5.5.1, take $w > 0$ (the integral is an even function of w),

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos wx\, dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{iwx}\, dx \qquad \left(\text{because } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \sin wx\, dx = 0\right)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-iw)^2-\frac{1}{2}w^2}\, dx\\ &= e^{-\frac{1}{2}w^2}\frac{1}{\sqrt{2\pi}}\overbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-iw)^2}\, dx}^{=J} = \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}}J.\end{aligned}$$

To evaluate J, consider the integral

$$I = \int_{\gamma_R} e^{-\frac{1}{2}(z-iw)^2}\, dz,$$

where γ_R is a rectangualr contour as in Figure 5.33, with length of the vertical sides equal to w. By Cauchy's theorem, $I = 0$ for all R.

Let I_j denote the integral on γ_j (see Example 5.5.1). Using the estimate in Example 5.5.1, we see that I_2 and I_4 tend to 0 as $R \to \infty$. On γ_3, $z = z + iw$, where x varies from R ro $-R$:

$$\int_{R}^{-R} e^{-\frac{1}{2}(x+iw-iw)^2}\, dx = -\int_{-R}^{R} e^{-\frac{1}{2}x^2}\, dx,$$

and this tends to $-\sqrt{2\pi}$ as $R \to \infty$, by (1), Sec. 5.4. Since $I = 0$ for all R, it follows that

$$J = \lim_{R\to\infty} I_{\gamma_1} = -\lim_{R\to\infty} I_{\gamma_3} = \sqrt{2\pi}.$$

Consequently,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos wx\, dx = e^{-\frac{w^2}{2}}$$

for all $w > 0$. Since the integral is even in w, the formula holds for $w < 0$. For $w = 0$ the formula follows from (5.5.1).

5. (a) Let $f(x) = \cos 2x$, then $f'(x) = -2\sin 2x$ and $f''(x) = -4\cos 2x$. If $x \in (0,\ \frac{\pi}{4})$, then $f''(x) < 0$ and the graph concaves down. So any chord joining two points on the graph of $y = \cos 2x$ above the interval $(0,\ \frac{\pi}{4})$ lies under the graph of $y = \cos 2x$. take the two points on the graph, $(0,\ 1)$ and $(\frac{\pi}{4},\ 0)$. The equation of the line joining them is $y = -\frac{4}{\pi}x + 1$. Since it is under the graph of $y = \cos 2x$ for $x \in (0, \frac{\pi}{4})$, we obtain

$$-\frac{4}{\pi}x + 1 \le \cos 2x \qquad \text{for } 0 \le x \le \frac{\pi}{4}.$$

(b) Let I_j denote the integral of e^{-z^2} over the path γ_j in Figure 5.45. Since e^{-z^2} is entire, by Cauchy's theorem, $I_1 + I_2 + I_3 = 0$.
(c) For I_1, $z = x$,

$$I_1 = \int_0^R e^{-x^2} dx \to \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \text{as } R \to \infty,$$

by (5.5.1). On γ_2, $z = Re^{i\theta}$ $z^2 = R^2(\cos 2\theta + i\sin 2\theta)$,

$$\left|e^{-z^2}\right| = \left|e^{-R^2(\cos 2\theta + i\sin 2\theta)}\right| = e^{-R^2\cos 2\theta} \leq e^{-R^2(1-\frac{4}{\pi}\theta)}.$$

Parametrize the integral I_2 and estimate:

$$\begin{aligned}
|I_2| &= \left|\int_0^{\frac{\pi}{4}} e^{-R^2 e^{2i\theta}} Rie^{i\theta} d\theta\right| \\
&\leq R\int_0^{\frac{\pi}{4}} \left|e^{-R^2 e^{2i\theta}} e^{i\theta}\right| d\theta \leq R\int_0^{\frac{\pi}{4}} e^{-R^2(1-\frac{4}{\pi}\theta)} d\theta \\
&\leq Re^{-R^2}\int_0^{\frac{\pi}{4}} e^{R^2\frac{4}{\pi}\theta} d\theta \\
&= Re^{-R^2}\frac{1}{R^2}\frac{\pi}{4}e^{R^2\frac{4}{\pi}\theta}\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4R}e^{-R^2}\left[e^{R^2} - 1\right],
\end{aligned}$$

which tends to 0 as $R \to \infty$.
(d) On γ_3, $z = xe^{i\frac{\pi}{4}}$, where x varies from R to 0, $dz = e^{i\frac{\pi}{4}}dx$. So

$$\begin{aligned}
I_3 &= -e^{i\frac{\pi}{4}}\int_0^R e^{-x^2 e^{i\frac{\pi}{2}}} dx = -e^{i\frac{\pi}{4}}\int_0^R e^{-x^2 i} dx \\
&= -e^{i\frac{\pi}{4}}\int_0^R (\cos x^2 - i\sin x^2) dx.
\end{aligned}$$

As $R \to \infty$, I_3 converges to

$$-e^{i\frac{\pi}{4}}\int_0^\infty (\cos x^2 - i\sin x^2) dx.$$

(e) Let $R \to \infty$ in the sum $I_1 + I_2 + I_3 = 0$ and get

$$\begin{gathered}
\tfrac{\sqrt{\pi}}{2} - e^{i\frac{\pi}{4}} \textstyle\int_0^\infty (\cos x^2 - i\sin x^2)dx = 0; \\
e^{i\frac{\pi}{4}} \textstyle\int_0^\infty (\cos x^2 - i\sin x^2)dx = \tfrac{\sqrt{\pi}}{2}; \\
\textstyle\int_0^\infty (\cos x^2 - i\sin x^2)dx = \tfrac{\sqrt{\pi}}{2}e^{-i\frac{\pi}{4}}; \\
\textstyle\int_0^\infty (\cos x^2 - i\sin x^2)dx = \tfrac{\sqrt{\pi}}{2}\left(\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2}\right).
\end{gathered}$$

The desired result follows upon taking real and imaginary parts.

9. We will integrate the function $f(z) = \frac{1}{z(z-1)\sqrt{z-2}}$ around the contour in Figure 5.48. Here $\sqrt{z-2}$ is defined with the branch of the logarithm with a branch cut on the positive real axis. It is mulitple valued on the semi-axis $x > 2$. Aproaching the real axis from above and to right of 2, we have $\lim_{z\to x}\sqrt{z-2} = \sqrt{x-2}$. Aproaching the real axis from below and to right of 2, we have $\lim_{z\to x}\sqrt{z-2} = -\sqrt{x-2}$. Write

$$I = I_1 + I_2 + I_3 + I_4,$$

where I_1 is the integral over the small circular part; I_2 is the integral over the interval above the x-axis to the right of 2; I_3 is the integral over the larger circular path; I_4 is the integral over the interval (neg. orientation) below the x-axis to the right of 2. We have $I_1 \to 0$ as $r \to 0$ and $I_3 \to 0$ as $R \to \infty$. (See Example 5.5.3 for similar details.) We have $I_2 \to I$ as $R \to \infty$ and $I_4 \to I$ as $R \to \infty$, where I is the desired integral. So

$$\begin{aligned}
2I &= 2\pi i\,[\operatorname{Res}(0) + \operatorname{Res}(1)]\,; \\
\operatorname{Res}(0) &= \lim_{z\to 0}\frac{1}{(z-1)\sqrt{z-2}} = -\frac{1}{\sqrt{-2}} \\
&= -\frac{1}{e^{\frac{1}{2}\log_0(-2)}} = -\frac{1}{e^{\frac{1}{2}(\ln 2 + i\pi)}} \\
&= -\frac{1}{i\sqrt{2}} = \frac{i}{\sqrt{2}}; \\
\operatorname{Res}(1) &= \frac{1}{\sqrt{-1}} = -i; \\
I &= i\pi\left[\frac{i}{\sqrt{2}} - i\right] = \pi\left[1 - \frac{1}{\sqrt{2}}\right].
\end{aligned}$$

13. (a) In Figure 5.51, let
γ_1 denote the small circular path around 0 (negative direction);
γ_2 the line segment from r to $1-r$, above the x-axis (positive direction);
γ_3 the small semi-circular path around 1 above the x-axis (negative direction);
γ_4 the line segment from $1+r$ to $R-1$, above the x-axis (positive direction);
γ_5 the large circular path around 0 (positive direction);
γ_6 the line segment from R to $1+r$, below the x-axis (negative direction);
γ_7 the small semi-circular path around 1 below the x-axis (negative direction);
γ_8 the line segment from $1-r$ to r, below the x-axis (negative direction).

We integrate the function

$$f(z) = \frac{z^p}{z(1-z)}$$

on the contour γ, where $z^p = e^{p\log_0 z}$ (branch cut along positive x-axis). By Cauchy's theorem,

$$\int_\gamma f(z)\,dz = 0 \quad \Rightarrow \quad \sum_{j=1}^{8} I_j = 0.$$

Review the integrals I_3, I_4, I_7, and I_8 from Example 5.5.3, then you can show in a similar way that I_1, I_3, and I_7 tend to 0 as $r \to 0$. Also $I_5 \to 0$ as $R \to \infty$. We will give some details. For I_1,

$$\begin{aligned}
|I_1| &= \left|\int_{\gamma_1}\frac{z^p}{z(1-z)}dz\right| \\
&= 2\pi r\frac{r^p}{r(1-r)} = 2\pi\frac{r^p}{(1-r)} \to 0 \text{ as } r \to 0.
\end{aligned}$$

For I_2 and I_5, we have

$$I_2 + I_5 \to \int_0^\infty \frac{x^p}{x(1-x)}\,dx, \text{ as } r \to 0 \text{ and } R \to \infty.$$

For I_3 and I_6, we have

$$I_3 + I_6 \to -\int_0^\infty \frac{e^{p\log_0 x}}{x(1-x)}\,dx = -e^{2p\pi i}\int_0^\infty \frac{x^p}{x(1-x)}\,dx, \text{ as } r \to 0 \text{ and } R \to \infty.$$

To evaluate I_4 and I_7, we use a trick that will allow us to apply Lemma 5.4.3. Note that on γ_4, $\log_0 z = \operatorname{Log} z$, and on γ_7, $\log_0 z = \log_{\frac{\pi}{2}} z$. This allows us to replace $\log_0$ by a branch of the log which is analytic in a neighborhood of the contour of integration and ths allows us to apply Lemma 5.4.3. According to this lemma, as $r \to 0$,

$$I_4 = \int_{\gamma_4} \frac{e^{p\operatorname{Log} z}}{z(1-z)} \to i\pi e^{p\operatorname{Log}(1)} = i\pi;$$

and

$$I_7 = \int_{\gamma_4} \frac{e^{p\log_{\frac{\pi}{2}} z}}{z(1-z)} \to i\pi e^{p\log_{\frac{\pi}{2}}(1)} = i\pi e^{2\pi p i}.$$

So as $r \to 0$,

$$I_4 + I_7 = \int_{\gamma_4} \frac{e^{p\operatorname{Log} z}}{z(1-z)} \to i\pi(1 + e^{2\pi p i}).$$

Adding the integrals together and then taking limits, we get

$$\begin{gathered}(1 - e^{2p\pi i})I + i\pi(1 + e^{2\pi p i}) = 0\\ I = -i\pi\tfrac{1+e^{2\pi p i}}{1-e^{2\pi p i}}\\ I = -i\pi\tfrac{e^{-\pi p i}+e^{\pi p i}}{e^{-\pi p i}-e^{\pi p i}} = \pi\cot p\pi.\end{gathered}$$

(b) Use $x = e^t$, do the substitution, then replace t by x, and get, from (a),

$$\text{P.V.}\int_{-\infty}^\infty \frac{e^{px}}{1-e^x}\,dx = \pi\cot p\pi \qquad (0 < p < 1).$$

(c) Change variables $x = 2u$, $dx = 2\,du$, then

$$\begin{aligned}\pi\cot p\pi &= 2\text{P.V.}\int_{-\infty}^\infty \frac{e^{2pu}}{1-e^{2u}}\,du\\ &= 2\text{P.V.}\int_{-\infty}^\infty \frac{e^{2pu}e^{-u}}{e^{-u}-e^{u}}\,du = -\text{P.V.}\int_{-\infty}^\infty \frac{e^{(2p-1)u}}{\sinh u}\,du\\ -\pi\cot\left(\pi\frac{w+1}{2}\right) &= \text{P.V.}\int_{-\infty}^\infty \frac{e^{wu}}{\sinh u}\,du \qquad (w = 2p-1).\end{aligned}$$

But

$$-\cot\left(\pi\frac{w+1}{2}\right) = \tan\frac{\pi w}{2},$$

so

$$\text{P.V.}\int_{-\infty}^\infty \frac{e^{wu}}{\sinh u}\,du = \tan\frac{\pi w}{2}.$$

Replace u by x and get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{wx}}{\sinh x}\, dx = \pi \tan \frac{\pi w}{2}.$$

(d) If $|a| < b$, take $t = bx$ and $w = a$, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh bx}\, dx = \frac{\pi}{b} \tan \frac{\pi a}{2b}.$$

(e) Replace a by $-a$ in (d) and get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-ax}}{\sinh bx}\, dx = -\frac{\pi}{b} \tan \frac{\pi a}{2b}.$$

Subtract from (d) and divide by 2:

$$\int_{-\infty}^{\infty} \frac{\sinh ax}{\sinh bx}\, dx = \frac{\pi}{b} \tan \frac{\pi a}{2b} \quad (b > |a|).$$

Note that the integral is convergent so there is no need to use the principal value.

17. We use a contour like the one in Figure 5.34. Let
γ_1 denote the small circular path around 0 (negative direction);
γ_2 the line segment from r to R, above the x-axis (positive direction);
γ_3 the large circular path around 0 (positive direction);
γ_4 the line segment from R to r, below the x-axis (negative direction). We integrate the function

$$f(z) = \frac{\sqrt{z}}{z^2 + z + 1}$$

on the contour γ, where $\sqrt{z} = e^{\frac{1}{2}\log_0 z}$ (branch cut along positive x-axis). By the residue theorem,

$$\int_\gamma f(z)\, dz = 2\pi i \sum_j \text{Res}\,(f,\ z_j) \quad \Rightarrow \quad \sum_{j=1}^{4} I_j = 2\pi i \sum_j \text{Res}\,(f,\ z_j),$$

where the sum is over all the residues of f in the region inside γ. The poles of f in this region are at the roots of $z^2 + z + 1 = 0$ or

$$z = \frac{-1 \pm \sqrt{-3}}{2}; \quad z_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \quad z_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}.$$

We have

$$|z_1| = \sqrt{(\tfrac{-1}{2})^2 + (\tfrac{\sqrt{3}}{2})^2} = 1, \quad z_1 = e^{i\frac{2\pi}{3}}$$
$$\log_0(z_1) = \ln|z_1| + i \arg{}_0(z_1) = 0 + i\tfrac{2\pi}{3} = i\tfrac{2\pi}{3}$$
$$\text{Res}\ (z_1) = \tfrac{\sqrt{z_1}}{2z_1+1} = \tfrac{e^{\frac{1}{2}\log_0(z_1)}}{2z_1+1}$$
$$\text{Res}\ (z_1) = \tfrac{e^{i\frac{\pi}{3}}}{2e^{i\frac{2\pi}{3}}+1}.$$

Similarly,

$$|z_2| = \sqrt{(\tfrac{-1}{2})^2 + (\tfrac{-\sqrt{3}}{2})^2} = 1, \quad z_2 = e^{i\frac{4\pi}{3}}$$
$$\log_0(z_2) = \ln|z_2| + i\arg{}_0(z_2) = 0 + i\tfrac{4\pi}{3} = i\tfrac{4\pi}{3}$$
$$\text{Res }(z_2) = \tfrac{\sqrt{z_2}}{2z_2+1} = \tfrac{e^{\frac{1}{2}\log_0(z_2)}}{2z_2+1}$$
$$\text{Res }(z_2) = \tfrac{e^{i\frac{2\pi}{3}}}{2e^{i\frac{4\pi}{3}}+1}.$$

Let us now compute the integrals. For I_1,

$$\begin{aligned} |I_1| &= \left|\int_{\gamma_1} \frac{\sqrt{z}}{z^2+z+1} dz\right| \\ &= 2\pi r \frac{\sqrt{r}}{1-r^2-r} \to 0 \text{ as } r \to 0. \end{aligned}$$

For I_2, we have

$$I_2 \to \int_0^\infty \frac{\sqrt{x}}{x^2+x+1}\,dx = I, \text{ as } r \to 0 \text{ and } R \to \infty.$$

A simple estimate shows that $I_3 \to 0$ as $R \to \infty$. For I_4, $\sqrt{z} = e^{\frac{1}{2}(\ln|z|+2\pi i)} = \sqrt{x}e^{i\pi}$. So

$$I_4 \to -\int_0^\infty \frac{e^{\frac{1}{2}\log_0 x}}{x^2+x+1}\,dx = -e^{\pi i}\int_0^\infty \frac{\sqrt{x}}{x^2+x+1}\,dx = I, \text{ as } r \to 0 \text{ and } R \to \infty.$$

Adding the integrals together and then taking limits, we get

$$\begin{aligned} 2I &= 2\pi i\left(\frac{e^{i\frac{\pi}{3}}}{2e^{i\frac{2\pi}{3}}+1} + \frac{e^{i\frac{2\pi}{3}}}{2e^{i\frac{4\pi}{3}}+1}\right) \\ I &= \pi i\left(\frac{e^{i\frac{\pi}{3}}}{2e^{i\frac{2\pi}{3}}+1} + \frac{e^{i\frac{2\pi}{3}}}{2e^{i\frac{4\pi}{3}}+1}\right) \\ &= \pi i\left(\frac{2e^{i\frac{5\pi}{3}} + e^{i\frac{\pi}{3}} + 2e^{i\frac{4\pi}{3}} + e^{i\frac{2\pi}{3}}}{4e^{i\frac{6\pi}{3}} + 2e^{i\frac{2\pi}{3}} + 2e^{i\frac{4\pi}{3}} + 1}\right) = \frac{\pi}{\sqrt{3}}. \end{aligned}$$

Use

$$\begin{aligned} e^{i\frac{\pi}{3}} &= \frac{1}{2} + \frac{i}{2}\sqrt{3}, & e^{i\frac{2\pi}{3}} &= -\frac{1}{2} + \frac{i}{2}\sqrt{3}, \\ e^{i\frac{3\pi}{3}} &= -1, & e^{i\frac{4\pi}{3}} &= -\frac{1}{2} - \frac{i}{2}\sqrt{3}, \\ e^{i\frac{5\pi}{3}} &= \frac{1}{2} - \frac{i}{2}\sqrt{3}, & e^{i\frac{6\pi}{3}} &= 1 \end{aligned}$$

21. (a) Letting $\gamma(t) = \sqrt{\alpha}(t-\beta)$, $a \le t \le b$, we have $\gamma'(t) = \sqrt{\alpha}$, and

$$\frac{1}{\sqrt{\alpha}}\int_\gamma e^{-z^2}\,dz = \frac{1}{\sqrt{\alpha}}\int_a^b e^{-\left(\sqrt{a}(t-\beta)\right)^2}\sqrt{\alpha}\,dt = \int_a^b e^{-\alpha(t-\beta)^2}\,dt.$$

Since $\operatorname{Re}\alpha > 1$, we must have $|\operatorname{Arg}\alpha| < \frac{\pi}{2}$, and since $\operatorname{Arg}\sqrt{\alpha} = \frac{1}{2}\operatorname{Arg}\alpha$, $|\operatorname{Arg}\sqrt{\alpha}| < \frac{\pi}{4}$.
(b) Let $\varepsilon = \frac{1}{2}\left(\frac{\pi}{4} - \operatorname{Arg}\sqrt{\alpha}\right)$. Then $\varepsilon > 0$ by (a) and $\frac{\pi}{4} - \varepsilon > \operatorname{Arg}\sqrt{\alpha}$. Thus, the line at angle $\theta = \operatorname{Arg}\sqrt{\alpha}$ must eventually pass under the line at angle $\theta = \frac{\pi}{4} - \varepsilon$ in the right half-plane, and symmetrically, the other end of the line must eventually pass above the line at angle $\theta = -\frac{3\pi}{4} - \varepsilon$ in the left half-plane.
(c) We have that

$$\left|e^{-z^2}\right| = e^{\operatorname{Re}\left(-R^2 e^{2i\theta}\right)} = e^{-R^2\cos(2\theta)}.$$

Since we are letting $R \to \infty$, we may assume R is sufficiently large as required in (b). Then, on γ_2, we have that $z = Re^{i\theta}$. By Figure 5.53, $0 \le \theta$, and by (b), $\theta \le \frac{\pi}{4} - \varepsilon < \frac{\pi}{4}$. Then $0 \le 2\theta < \frac{\pi}{2}$, and $\cos(2\theta) > 0$. Since $\ell(\gamma_2) < \frac{\pi}{4}R$, by the ML-inequality,

$$|I_2| \le \frac{\pi}{4} R e^{-R^2\cos(2\theta)} \to 0 \quad \text{as} \quad R \to \infty.$$

Similarly, $|I_4| \to 0$ as $R \to \infty$.
(d) Since the integrand is entire,

$$\int_\gamma e^{-z^2}\, dz = 0 = I_1 + I_2 + I_3 + I_4.$$

Letting $R \to \infty$ and using (a) gives us

$$\sqrt{\alpha}\int_{-\infty}^{\infty} e^{-\alpha(t-\beta)^2}\, dt + \int_{\infty}^{-\infty} e^{-x^2}\, dx = 0.$$

But $\int_{\infty}^{-\infty} e^{-x^2}\, dx = -\int_{-\infty}^{\infty} e^{-x^2}\, dx = -\sqrt{\pi}$. Solving for the desired integral gives us

$$\int_{-\infty}^{\infty} e^{-\alpha(t-\beta)^2}\, dt = \sqrt{\frac{\pi}{\alpha}}.$$

Solutions to Exercises 5.6

1. Note that $f(z) = \frac{1}{z^2+9}$ has two simples poles at $\pm 3i$. Since $f(z)$ does not have poles on the integers, we may apply Proposition 1 and get

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{k^2+9} &= -\pi \left[\operatorname{Res}\left(f(z)\cot \pi z,\ 3i\right) + \operatorname{Res}\left(f(z)\cot \pi z,\ -3i\right)\right] \\
&= -\pi \left[\lim_{z\to 3i} \frac{\cot \pi z}{z+3i} + \lim_{z\to -3i} \frac{\cot \pi z}{z-3i}\right] \\
&= -\pi \left[\frac{1}{6i}\cot(3\pi i) - \frac{1}{6i}\cot(-3\pi i)\right] \\
&= \frac{\pi}{3}\left[i\cot(3\pi i)\right] = \frac{\pi}{3}\coth(3\pi),
\end{aligned}$$

because $\cot(iz) = -i\coth(z)$. You can prove the last identity by using (25) and (26) of Section 1.6.

5. Reason as in Exercise 1 with $f(z) = \frac{1}{4z^2-1}$, which has two simples poles at $\pm\frac{1}{2}$. Since $f(z)$ does not have poles on the integrers, we may apply Proposition 1 and get

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} &= -\pi \left[\operatorname{Res}\left(f(z)\cot \pi z,\ \frac{1}{2}\right) + \operatorname{Res}\left(f(z)\cot \pi z,\ -\frac{1}{2}\right)\right] \\
&= -\pi \left[\lim_{z\to \frac{1}{2}} \frac{\cot \pi z}{4(z+\frac{1}{2})} + \lim_{z\to -\frac{1}{2}} \frac{\cot \pi z}{4(z-\frac{1}{2})}\right] \\
&= -\frac{\pi}{2}\left[\cot\left(\frac{\pi}{2}\right) - \cot\left(-\frac{\pi}{2}\right)\right] = 0,
\end{aligned}$$

because $\cot\left(\frac{\pi}{2}\right) = 0$.

9. Reason as in Exercise 1 with $f(z) = \frac{1}{(z-2)(z-1)+1}$, which has two simples poles where $z^2 - 3z + 3 = 0$ or

$$z = \frac{3 \pm \sqrt{-3}}{2}; \quad z_1 = \frac{3+i\sqrt{3}}{2}, \quad z_2 = \frac{3-i\sqrt{3}}{2}.$$

Since $f(z)$ does not have poles on the integrers, we may apply Proposition 1 and get

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-2)(k-1)+1} = -\pi \left[\operatorname{Res}\left(f(z)\cot \pi z,\ z_1\right) + \operatorname{Res}\left(f(z)\cot \pi z,\ z_2\right)\right]$$

Let's compute:

$$\begin{gathered}
\operatorname{Res}(z_1) = \lim_{z\to z_1} \frac{\cot \pi z}{(z-z_2)} = \frac{\cot \pi z_1}{(z_1-z_2)} \\
z_1 - z_2 = i\sqrt{3}, \\
\cot(\pi z_1) = \cot\left(\pi\frac{3+i\sqrt{3}}{2}\right) = -\tan\left(i\pi\frac{\sqrt{3}}{2}\right) = -i\tanh\left(\pi\frac{\sqrt{3}}{2}\right).
\end{gathered}$$

(Prove and use the identities $\cot\left(z + \frac{3\pi}{2}\right) = -\tan(z)$ and $\tan(iz) = i \tanh z$.) So

$$\operatorname{Res}(z_1) = \frac{-i \tanh\left(\frac{\sqrt{3}}{2}\pi\right)}{i\sqrt{3}} = -\frac{\tanh\left(\frac{\sqrt{3}}{2}\pi\right)}{\sqrt{3}}.$$

A similar computation shows that $\operatorname{Res}(z_2) = \operatorname{Res}(z_1)$, hence

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-2)(k-1)+1} = \frac{2\pi}{\sqrt{3}} \tanh\left(\frac{\sqrt{3}}{2}\pi\right)$$

13. In the proof of Proposition 5.6.2, we apply (5.6.1) to all $z \in \mathbb{Z}$ except for $z = 0$, since f may no longer be analytic at $z = 0$. Thus, we must include $z_j = 0$ in the right sum of (5.6.5) rather than in the left sum, which now excludes $k = 0$, giving us (5.6.6).

17. (a) Apply the result of Execrise 13 with $f(z) = \frac{1}{z^{2n}}$:

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^{2n}} = -\pi \operatorname{Res}\left(\frac{\cot(\pi z)}{z^{2n}}, 0\right),$$

because f has only one pole of order $2n$ at 0. The sum on the left is even, so

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = -\frac{\pi}{2} \operatorname{Res}\left(\frac{\cot(\pi z)}{z^{2n}}, 0\right).$$

(b) Recall the Taylor series expansion of $z \cot z$ from Exercise 31, Section 4.4,

$$\begin{aligned} z \cot z &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} z^{2k} \\ (\pi z) \cot(\pi z) &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k} \pi^{2k}}{(2k)!} z^{2k} \\ \cot(\pi z) &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k} \pi^{2k-1}}{(2k)!} z^{2k-1}. \end{aligned}$$

So

$$\frac{\cot(\pi z)}{z^{2n}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k} \pi^{2k-1}}{(2k)!} z^{+2k-2n-1}.$$

The residue at 0 is a_{-1}, the coefficient of $\frac{1}{z}$, which is obtained from the series above when $2k - 2n - 1 = -1$ or $n = k$. Hence

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^{2n}}, 0\right) = (-1)^n \frac{2^{2n} B_{2n} \pi^{2n-1}}{(2n)!}.$$

Using (a), and the fact that the summand is even, we get

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!}.$$

Solutions to Exercises 5.7

1. The roots of the polynomial are easy to find using the quadratic formula:

$$z^2 + 2z + 2 = 0 \quad \Rightarrow \quad z = -1 \pm i.$$

Thus no roots are in the first quadrant. This, of course, does not answer the exercise. We must arrive at this answer using the method of Example 1, with the help of the argument principle.

First, we must argue that f has no roots of the positive $x-axis$. This is clear, because if $x > 0$ then $x^2 + x + 2 > 2$ and so it cannot possibly be equal to 0. Second, we must argue that there are no roots on the upper imaginary axis. $f(iy) = -y^2 + 2y + 2 = 2 - y^2 + 2y$. If $y = 0$, $f(0) = 2 \neq 0$. If $y > 0$, then $\operatorname{Im}(f(y)) = 2y \neq 0$. In all cases, $f(iy) \neq 0$ if $y \geq 0$.

The number of zeros of the polynomial $f(z) = z^2 + 2z + 2$ is equal to the number of times the image of σ_R wraps around the origin, where γ_R is the circular path in the first quadrant, in Fig. 4. This path consists of the interval $[0,\ R]$, the circular arc σ_R, and the interval on the imaginary axis from iR to 0. To find the image on γ_R, we consider the image of each component separately.

Since $f(x)$ is real for real x, we conclude that the image of the interval $[0,\ R]$ is also an interval, and it is easy to see that this interval is $[2,\ R^2 + 2R + 2]$. So its initial point is $w_0 = 2$ and its terminal point is $w_1 = R^2 + 2R + 2$.

The image of the arc σ_R starts at the point $w_1 = R^2 + 2R + 2$ and ends at $f(iR) = -R^2 + 2iR + 2 = w_2$, which is the image of the terminal point of σ_R. We have $\operatorname{Im}(w_2) = 2R$ and $\operatorname{Re}(w_2) = 2 - R^2 < 0$ if R is very large. Hence the point $f(w_2)$ is in the second quadrant. Also, for very large R, and $|z| = R$, the mapping $z \mapsto f(z)$ is approximately like $z \mapsto z^2$. So $f(z)$ takes σ_R and maps it "approximately" to the semi-circle (the map $w = z^2$ doubles the angles), with initial point w_1 and terminal point w_2.

We now come to the third part of the image of γ_R. We know that it starts at w_2 and end at w_0. As this image path go from w_2 to w_0, does it wrap around zero or not? To answer this question, we consider $f(iy) = 2 - R^2 + 2iy$. Since $\operatorname{Im}(f(iy) > 0$ if $y > 0$, we conclude that the image point of iy remains in the upper half-plane as it moves from w_2 to w_0. Consequently, the image curve does not wrap around 0; and hence the polynomial has no roots in the first quadrant-as expected.

5. We follow the steps in the solution of Exercise 1, but here the roots of $z^4 + 8z^2 + 16z + 20 = 0$ are not so easy to find, so we will not give them.

Argue that f has no roots of the positive $x - axis$. This is clear, because if $x > 0$ then $x^4 + 8x^2 + 16z + 20 > 20$ and so it cannot possibly be equal to 0. Second, we must argue that there are no roots on the upper imaginary axis. $f(iy) = y^4 - 8y^2 + 16iy + 20 = y^4 - 8y^2 + 20 - 8iy$. If $y = 0$, $f(0) = 20 \neq 0$. If $y > 0$, then $\operatorname{Im}(f(y)) = -8y \neq 0$. In all cases, $f(iy) \neq 0$ if $y \geq 0$.

The number of zeros of the polynomial $f(z) = z^4 + 8z^2 + 16z + 20$ is equal to the number of times the image of σ_R wraps around the origin, where γ_R is the circular path in the first quadrant, in Fig. 4. This path consists of the interval $[0,\ R]$, the circular arc σ_R, and the interval on the imaginary axis from iR to 0. To find the image on γ_R, we consider the image of each component separately.

Since $f(x)$ is real for real x, we conclude that the image of the interval $[0,\ R]$ is also an interval, and it is easy to see that this interval is $[20,\ R^4+8R^2+16R+20]$. So its initial point is $w_0 = 2$ and its terminal point is $w_1 = R^4+8R^2+16R+20$.

The image of the arc σ_R starts at the point $w_1 = R^4+8R^2+16R+20$ on the real axis and ends at $f(iR) = R^4-8R^2+20+16iR = w_2$, which is the image of the terminal point of σ_R. We have $\operatorname{Im}(w_2)) = 16R > 0$ and $\operatorname{Re}(w_2) = R^4-8R^2+20 > 0$ if R is very large. Hence the point $f(w_2)$ is in the first quadrant. Also, for very large R, and $|z| = R$, the mapping $z \mapsto f(z)$ is approximately like $z \mapsto z^2$. So $f(z)$ takes σ_R and maps it "approximately" to a circle (the map $w = z^4$ multiplies angles by 4), with initial point w_1 and terminal point w_2. So far, the image of $[0,\ R]$ and σ_R wraps one around the origin.

We now come to the third part of the image of γ_R. We know that it starts at w_2 and end at w_0. As this image path go from w_2 to w_0, does it close the loop around 0 or does it unwrap it? To answer this question, we consider $f(iy) = R^4-8R^2+20+16iy$. Since $\operatorname{Im}(f(iy) = 16y > 0$ if $y > 0$, we conclude that the image point of iy remains in the upper half-plane as it moves from w_2 to w_0. Consequently, the image curve wraps around 0; and hence the polynomial has one root in the first quadrant.

9. Apply Rouché's theorem with $f(z) = 11$, $g(z) = 7z^3+3z^2$. On $|z| = 1$, $|f(z)| = 11$ and $|g(z)| \le 7+3 = 10$. Since $|f| > |g|$ on $|z| = 1$, we conclude that $N(f) = N(f+g)$ inside $C_1(0)$. Since $N(f) = 0$ we conclude that the polynomial $7z^3+3z^2+11$ has no roots in the unit disk.

13. Apply Rouché's theorem with $f(z) = -3z$, $g(z) = e^z$. On $|z| = 1$, $|f(z)| = 3$ and

$$|g(z)| = \left|e^{\cos t + i\sin t}\right| = e^{\cos t} \le e.$$

Since $|f| > |g|$ on $|z| = 1$, we conclude that $N(f) = N(f+g)$ inside $C_1(0)$. Since $N(f) = 1$ we conclude that the function $e^z - 3z$ has one root in the unit disk.

17. Apply Rouché's theorem with $f(z) = 5$, $g(z) = z^5+3z$. On $|z| = 1$, $|f(z)| = 5$ and $|g(z)| \le 4$. Since $|f| > |g|$ on $|z| = 1$, we conclude that $N(f) = N(f+g)$ inside $C_1(0)$. Since $N(f) = 0$ we conclude that the function z^5+3z+5 has no roots in the unit disk. Hence $\frac{1}{z^5+3z+5}$ is analytic in the unit disk and by Cauchy's theorem

$$\int_{C_1(0)} \frac{dz}{z^5+3z+5} = 0.$$

21. (a) Let $g(z) = z$ and $f(z) = z^n - 1$. Then

$$\frac{1}{2\pi i}\int_{C_R(0)} z\frac{nz^{n-1}}{z^n-1}\,dz = \frac{1}{2\pi i}\int_{C_R(0)} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^n m(z_j)g(z_j) = \sum_{j=1}^n z_j,$$

where z_j are the roots of $f(z)$ and $m(z_j)$ are their orders. But $m(z_j) = 1$ since $f'(z_j) \ne 0$. The z_j are precisely the n roots of unity. Let $S = \sum_{j=1}^n z_j$. Let $\zeta = \frac{1}{z}$, $d\zeta = -\frac{1}{z^2}dz$, as z runs through $C_R(0)$ in the positive direction, ζ runs through $C_{1/R}(0)$ in the negative direction. Let $C_{1/R}(0)^*$ denote the circle with radius $1/R$ with negative orientation.

Then

$$\frac{1}{2\pi i}\int_{C_R(0)} z\frac{nz^{n-1}}{z^n-1}\,dz = \frac{1}{2\pi i}\int_{C_{\frac{1}{R}}(0)^*}\frac{n}{\zeta^n(\frac{1}{\zeta^n}-1)}\,\frac{d\zeta}{-\zeta^2} = \frac{n}{2\pi i}\int_{C_{\frac{1}{R}}(0)}\frac{d\zeta}{(1-\zeta^n)\zeta^2}.$$

(b) Evaluate the second integral in part (a) using Cauchy's generalized integral formula and conclude that

$$\frac{n}{2\pi i}\int_{-C_{\frac{1}{R}}(0)}\frac{d\zeta}{(1-\zeta^n)\zeta} = \frac{n}{2\pi i}\int_{-C_{\frac{1}{R}}(0)}\frac{(1-\zeta^n)^{-1}}{\zeta-0}d\zeta = n\frac{d}{d\zeta}\frac{1}{1-\zeta^n}\Big|_{\zeta=0} = \frac{-n^2\zeta^{n-1}}{(1-\zeta^n)^2}\Big|_{\zeta=0} = 0.$$

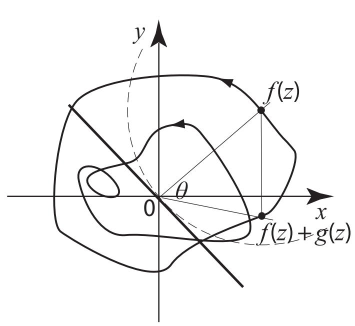
Using (a), we find $S=0$.

A different way to evaluate S is as follows: From (a),

$$\begin{aligned}
S &= \frac{n}{2\pi i}\int_{C_R(0)}\frac{z^n}{z^n-1}\,dz\\
&= \frac{n}{2\pi i}\int_{C_R(0)}\frac{z^n-1}{z^n-1}\,dz + \frac{n}{2\pi i}\int_{C_R(0)}\frac{1}{z^n-1}\,dz\\
&= \frac{n}{2\pi i}\int_{C_R(0)}1\cdot dz + \frac{n}{2\pi i}\int_{C_R(0)}\frac{1}{z^n-1}\,dz\\
&= 0+0=0.
\end{aligned}$$

The first integral is 0 by Cauchy's theorem. The second integral is zero because $C_R(0)$ contains all the roots of $p(z)=z^n-1$ (see Exercise 38, Sec. 3.4).

25. (i) Given z in the simple closed path C, consider the circle $T_{f(z)}$ centered at the point $f(z)$ with radius $|f(z)|$. Since $|g(z)| = |g(z)| < |f(z)|$ for $z\in C$, it follows that the distance $|g(z)|$ from $f(z)+g(z)$ to $f(z)$ is at most $|f(z)|$ and thus $f(z)+g(z)$ lies inside $T_{f(z)}$. The part of the image $f[C]$ that lies in the interior of the circle $T_{f(z)}$ is the image of a subarc γ_j as described in the discussion after Proposition 5.7.4. Since the interior of the circle $T_{f(z)}$ is a simply connected region that does not contain the origin, there is a branch of the logarithm in this region, which defines a branch of the argument. Thus there is an argument function defined on $T_{f(z)}$. Note now that in view of the picture below



the difference between the argument of $f(z)$ and the argument of $f(z)+g(z)$ is θ which is at most $\frac{\pi}{2}$ in absolute value. This proves that

$$\arg f(z) - \frac{\pi}{2} < \arg\big(f(z)+g(z)\big) < \arg f(z) + \frac{\pi}{2}.$$

(ii) As in the discussion after Proposition 5.7.4, we pick points $z_1, \dots, z_{n-1}$ (and we set $z_n = z_0$) such that the image of each subarc of C between $f(z_j)$ and $f(z_{j+1})$ $(j = 0, 1, \dots, n-1)$ is contained in a simply connected region and choose argument functions $\arg_{j+1}$ on this region such that

$$\arg_1 f(z_1) = \arg_2 f(z_1), \quad \arg_2 f(z_2) = \arg_3 f(z_2), \quad \dots \quad \arg_n f(z_n) = \arg_n f(z_0),$$

Then by part (i) we have

$$\arg_1 f(z_0) - \frac{\pi}{2} < \arg_1\big(f(z_0)+g(z_0)\big) < \arg_1 f(z_0) + \frac{\pi}{2}.$$

and also

$$\arg_n f(z_n) - \frac{\pi}{2} < \arg_n\big(f(z_n)+g(z_n)\big) < \arg_n f(z_n) + \frac{\pi}{2}.$$

which is the same as

$$\arg_n f(z_0) - \frac{\pi}{2} < \arg_n\big(f(z_0)+g(z_0)\big) < \arg_n f(z_0) + \frac{\pi}{2}.$$

Adding the preceding inequality to

$$-\arg_1 f(z_0) + \frac{\pi}{2} > -\arg_1\big(f(z_0)+g(z_0)\big) > -\arg_1 f(z_0) - \frac{\pi}{2}.$$

we obtain that

$$\arg_n f(z_0) - \arg_1 f(z_0) - \pi < \arg_n\big(f(z_0)+g(z_0)\big) - \arg_1\big(f(z_0)+g(z_0)\big) < \arg_n f(z_0) - \arg_1 f(z_0) + \pi$$

or equivalently

$$\Big|\Delta_C \arg\big(f+g\big) - \Delta_C \arg f\Big| < \pi.$$

(Here $\Delta_C \arg f = \arg_n f(z_0) - \arg_1 f(z_0)$.) Since the function Δ_C is integer-multiple of 2π, if follows that

$$\Delta_C \arg\big(f+g\big) - \Delta_C \arg f = 0$$

hence the number of zeros of f inside C is the same as that of $f+g$ inside C.

29. We are going to find the number zero of the the polynomial $p(z) = z^5 + z^4 + 6z^2 + 3z + 1$ in the unit disk by using Hurwitz's theorom. To do so, it suffices to show the next parts.
(a) Consider the polynomial $p_(z) = p(z) - \frac{1}{n}$ we will show that $p_n(z) \to p(z)$ converges uniformly on the closed unit disk. First of all, we say that

$$f_n \to f \quad \text{uniformly} \quad \Leftrightarrow \quad M_n = Max|f_n - f| \to 0 \quad \text{as} \quad n \to 0$$

Now let $p_n(z) = p(z) - \frac{1}{n}$. Then

$$M_n = Max|p_n(z) - p(z)| = \frac{1}{n}$$

Since $\lim_{n\to\infty}\frac{1}{n}=0$, we have that $p_n \to p$ converges uniformly.
(b) Let $p_n(z)=z^5+z^4+6z^2+3z+1-\frac{1}{n}$. Applying the Rouché's Theorem on $|z|=1$ with $f(z)=6z^2$, $g(z)=z^5+z^4+3z+1-\frac{1}{n}$, we get

$$|g(z)|=|z^5+z^4+3z+1-\frac{1}{n}|\leq|z|^5+|z|^4+3|z|+|1-\frac{1}{n}|<6=6|z|^2=|f(z)|$$

Since $|g(z)|<|f(z)|$ on $|z|=1$, we have that $N(f)=N(f+g)=N(p_n)$. But since $f(z)=6z^2$ has 2 zeros $f+g=p_n$ has 2 zeros.
(c) In part (a), we showed that $p_n\to p$ converges uniformly. Also, it is clear that $p(z)=z^5+z^4+6z^2+3z+1$ is not identically zero. Then, by Hurwitz's Theorem p_n and p has the same number of zeros. But, by part (b), we showed that p_n has two zeros. Therefore, p has two zeros.

33. For this question we are going to apply Lagrange Inversion formula. Consider the equation the is given as $z=a+we^z$, then we have

$$w=f(z)=(z-a)e^{-z}.$$

Let $z_0=a$, and evaluating this, we get

$$w_0=f(z_0=a)=(a-a)e^{-a}=0.$$

Also, it is clear that $f(z)=(z-a)e^{-z}$ is analytic at $z_0=a$ and $f'(z_0)\neq 0$. Indeed,

$$f'(z)=e^{-z}-(z-a)e^{-z}$$

$$\Rightarrow f'(a)=e^{-a}\neq 0.$$

Now we apply Lagrange Inversion formula. But first we define $\phi(z)$ that is given in with (5.7.13), we get

$$\phi(z)=\frac{z-z_0}{f(z)-w_0}.$$

Considering $z_0=a$ and $w_0=0$ in $\phi(z)$, we get

$$\phi(z)=\frac{z-a}{(z-a)e^{-z}}=e^z.$$

$$\Rightarrow [\phi(z)]^n=e^{nz}.$$

Now differentiate it $n-1$ times , we get

$$\frac{d}{dz}[\phi(z)]^n=ne^{nz}$$
$$\frac{d^2}{dz^2}[\phi(z)]^n=n^2e^{nz}$$
$$\vdots$$
$$\frac{d^{n-1}}{dz^{n-1}}[\phi(z)]^n=n^{n-1}e^{nz}.$$

Evaluating the last equation for $z_0 = a$, we get

$$\frac{d^{n-1}}{dz^{n-1}}[\phi(z)]^n\bigg|_{z_0=a} = n^{n-1}e^{an}.$$

Now if we plug all these values, and $z_0 = a$ and $w_0 = 0$ into the formula that is given by (5.7.14), we obtain

$$z = g(w) = z_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}[\phi(z)]^n\bigg|_{z_0=a}(w-0)^n \quad \text{or;}$$

$$z = g(w) = a + \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}e^{na}w^n.$$

Solutions to Exercises 6.1

1. (a) The function $u(x, y) = xy$ is harmonic on $\Omega = \mathbb{C}$, because

$$u_{xx} = 0,\ u_{yy} = 0, \text{and so } \Delta u = 0.$$

(b) To find the conjugate gradient of u, apply Lemma 6.1.7: for $z = x + iy$,

$$\phi(z) = u_x - iu_y = y - ix = iz.$$

Clearly, ϕ is analytic in Ω as asserted by Lemma 6.1.7

5. We will directly verify Laplace's equation. For $u = x^2 - y^2 + 2x - y$ we evaluate partial derivatives

$$u_{xx} = (u_x)_x = (2x+2)_x = 2$$

and

$$u_{yy} = (u_y)_y = (-2y-1)_y = -2.$$

Since $u_{xx} + u_{yy} = 2 - 2 = 0$ Laplace's equation is satisfied and u is a harmonic function.

9. Here we verify Laplace's equation again. For $u = \frac{1}{x+y}$ we evaluate partial derivatives

$$u_{xx} = (u_x)_x = \left(-\frac{1}{(x+y)^2}\right)_x = \frac{2}{(x+y)^3}$$

and

$$u_{yy} = (u_y)_y = \left(-\frac{1}{(x+y)^2}\right)_y = \frac{2}{(x+y)^3}.$$

Since $u_{xx} + u_{yy} = \frac{4}{(x+y)^3} \neq 0$, Laplace's equation is not satisfied and u is not a harmonic function.

13. To use Theorem 6.1.2, we can first notice that the following identity is true

$$(x + i\,y)^2 = x^2 + 2i\,xy + (iy)^2 = x^2 - y^2 + i\,2xy.$$

Using the polar form for $z = x + iy$ we have

$$e^{z^2} = e^{x^2-y^2+i\,2xy} = e^{x^2-y^2}(\cos(2xy) + i\,\sin(2xy)).$$

Since $u = \operatorname{Re}(e^{z^2})$ we conclude from Theorem 6.1.2 that u is harmonic.

17. To find a harmonic conjugate v for $u = x + 2y$, we use the Cauchy-Riemann equations as follows. We want $u + i\,v$ to be analytic. Hence v must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

Since $\frac{\partial u}{\partial x} = 1$, the first equation implies that

$$1 = \frac{\partial v}{\partial y}.$$

To get v we will integrate both sides of this equation with respect to y. Doing so we fix a value of x. Therefore the result of integration with respect to y is a function of y plus some constant which eventually may depend on x. Thus integrating with respect to y yields

$$v(x, y) = y + c(x),$$

where $c(x)$ is a function of x alone. Plugging this into the second equation in (1), we get

$$2 = -\big(0 + \frac{d}{dx}c(x)\big),$$

or equivalently

$$\frac{d}{dx}c(x) = -2.$$

If we integrate this equation with respect to x we get that $c(x) = -2x + C$ where C is any real constant. Substituting the expression for $c(x)$ into the expression (2) for v we get $v(x, y) = y + c(x) = y - 2x + C$. Now we check the answer by using Cauchy-Riemann equations. We have $u_x = 1$, $u_y = 2$, $v_x = -2$, and $v_y = 1$. Since $u_x = v_y$ and $u_y = -v_x$ we conclude that v is a harmonic conjugate of u.

21. By the Cauchy-Riemann equations, we have representations for the derivative as follows:

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

We know that $\phi = u_x - iu_y$ and that $-u_y = v_x$ which in turn implies that

$$f'(z) = u_x - iu_y = \phi.$$

25. This function is harmonic because $\Delta u = u_{xx} + u_{yy} = e^x \cos(y) - e^x \cos(y) = 0$. By the Maximum modulus principle Corollaries, u must attain its maximum and minimum values on $\partial\Omega$. Evaluating we will find that u will attain its maximum and minimum on the vertices of the square. The maximum will happen at π and the minimums will happen at $i\pi$ and $-i\pi$.

29. From the fact that u is harmonic, we have

$$u_{xx} + u_{yy} = 0.$$

Then, becasue u^2 is harmonic we can calculate that

$$(u^2)_{xx} = 2(uu_{xx} + u_x u_x)$$

and

$$(u^2)_{yy} = 2(uu_{yy} + u_y u_y).$$

Now plugging that into Laplace's equation we get

$$\begin{aligned} uu_{xx} + u_x u_x + uu_{yy} + u_y u_y &= 0 \\ u(u_{xx} + u_{yy}) + u_x^2 + u_y^2 &= 0 \\ u_x^2 + u_y^2 &= 0. \end{aligned}$$

Since u is real valued, we have that $u_x = u_y = 0$ which implies that u is constant.

33. Taking partial derivatives we find that

$$\partial_x(u(x, -y)) = (\partial_x u)(x, -y)$$

$$\begin{aligned} \partial_{xx}(u(x, -y)) &= -(\partial_{xx} u)(x, -y) \\ \partial_y(u(x, -y)) &= (\partial_y u)(x, -y) \\ \partial_{yy}(u(x, -y)) &= (\partial_{yy} u)(x, -y) \end{aligned}$$

using the fact that we will get a double negative in the double partial with respect to y. Thus we have that $\Delta u = (\partial_{xx} u)(x, -y) + (\partial_{yy} u)(x, -y) = 0$.

37. We show that if u and e^u are harmonic on a region Ω, then u is identically constant.

Let $\phi = e^u$. Then $\phi_x = e^u u_x$ and $\phi_{xx} = e^u (u_x)^2 + e^u u_{xx}$. Similarly, $\phi_y = e^u u_y$ and $\phi_{yy} = e^u (u_y)^2 + e^u u_{yy}$. Using $\Delta \phi = 0$ and $\Delta u = 0$, we get

$$e^u[(u_x)^2 + (u_y)^2] = 0 \Rightarrow (u_x)^2 + (u_y)^2 = 0 \Rightarrow u_x = 0 \text{ and } u_y = 0.$$

Hence u is constant.

Solutions to Exercises 6.2

1. (a) We have

$$u_x = a, u_{xx} = 0, u_y = o, u_{yy} = 0$$

which implies that $\Delta u = 0$.
(b) We have the data that

$$\begin{aligned} u(3,0) &= 10 \\ u(5,0) &= 40 \end{aligned}$$

Plugging this into our equation for u this gives us

$$\begin{aligned} 3a + b &= 10 \\ 5a + b &= 40. \end{aligned}$$

Solving this system of equations we get $u(x,y) = 15x - 35$.
(c) For the isotherms, we solve $u(x,y) = T$ which implies

$$x = \frac{T + 35}{15}.$$

To determine heat flow, we want to solve

$$v = \int_{z_0}^{z} -u_y dx + u_x dy = \int_0^1 15iydy.$$

Thus we get

$$v(x,y) = 15iy = c$$

for a constant c which gives

$$y = \frac{c}{15i}.$$

5.(a) Doing the differentiation we get first

$$2rr_x = 3x \implies r_x = \frac{x}{r}.$$

Differentiating a second time we get

$$\begin{aligned} r_{xx} = \frac{r - xr_x}{r^2} &= \frac{r - \frac{x^2}{r}}{r^2} \\ &= \frac{\frac{r^2}{r} - \frac{x^2}{r}}{r^2} \\ &= \frac{r^2 - x^2}{r^3} \\ &= \frac{x^2 + y^2 - x^2}{r^3} \\ &= \frac{y^2}{r^3} \end{aligned}$$

giving us the desired result.
(b) Taking the first derivative we get

$$\begin{aligned}\theta_x &= \arctan\left(\frac{y}{x}\right)\\ &= \frac{1}{1+\frac{y^2}{x^2}}\frac{-y}{x^2}\\ &= -\frac{y}{x^2+y^2}\\ &= -\frac{y}{r^2}.\end{aligned}$$

Differentiating a second time we get

$$\begin{aligned}\theta_{xx} &= \left(\frac{-y}{r^2}\right)_x\\ &= \frac{2yrr_x}{r_4}\\ &= \frac{2yr\frac{x}{r}}{r^4}\\ &= \frac{2xy}{r^4}\end{aligned}$$

giving us the desired result.
(c) Follow the same steps as above.
(d) Substituting what we found in the previous steps we will find that

$$\theta_{xx}+\theta_{yy} = \frac{2xy}{r^4} - \frac{2xy}{y^4} = 0$$

and

$$\theta_x r_x + \theta_y r_y = -\frac{y}{r^2}\frac{x}{r} + \frac{x}{r^2}\frac{y}{r} = 0$$

giving us the desired result.
(e) Using the chain rule in two dimensions and the fact that our function u is of the form $u(r,\theta) = r(x,y)\theta(x,y)$, we get that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

Differentiating again we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}.$$

Notice now that we have another chain rule for $\frac{\partial}{\partial x}\frac{\partial u}{\partial r}$ and $\frac{\partial}{\partial x}\frac{\partial x}{\partial \theta}$. Doing so gives us the desired result. Change x yo y to get similar results for the double partial with respect to y.

(f) Using the identity from (d), and the results from (a),(b), and (c) we get that

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 u}{\partial \theta^2}(\frac{\partial \theta}{\partial x})^2 + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 \\
&= \frac{\partial^2 u}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + \frac{\partial u}{\partial r}\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right] + \frac{\partial^2 u}{\partial \theta^2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right] \\
&= \frac{\partial^2 u}{\partial r^2}\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right] + \frac{\partial u}{\partial r}\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] + \frac{\partial^2 u}{\partial \theta^2}\left[\left(\frac{-y}{r^2}\right)^2 + \left(\frac{x}{r^2}\right)^2\right] \\
&= \frac{\partial^2 u}{\partial r^2}\left[\frac{1}{r^2}(x^2+y^2)\right] + \frac{\partial u}{\partial r}\left[\frac{1}{r^3}(x^2+y^2) + \frac{\partial^2 u}{\partial \theta^2}\left[\frac{1}{r^4}(x^2+y^2)\right]\right. \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.
\end{aligned}$$

9.We want to solve the system of equations

$$c_1 ln(R_1) + c_2 = T_1$$

$$c_1 ln(R_2) + c_2 = T_2.$$

Subtracting the first equation from the second gives us

$$c_1(\ln(R_2) - \ln(R_1)) = T_2 - T_1$$

which implies $c_1 = \frac{T_2 - T_1}{ln(\frac{R_2}{R_1})}$. Substituting this value into the first equation we find

$$\frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)}\ln(R_1) + c_2 = T_1$$

which implies that $c_2 = T_1 - \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)}\ln(R_1)$. Thus we have that

$$\begin{aligned}
u(r) &= c_1 \ln(r) + c_2 \\
&= \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)}\ln(r) + T_1 - \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)}\ln(R_1) \\
&= T_1 + \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)}(\ln(r) - \ln(R_1)) \\
&= T_1 + (T_2 - T_1)\frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}.
\end{aligned}$$

This gives us that the polar form of the Laplacian is

$$\Delta u = u_{rr} + \frac{1}{r}u_r = 0$$

because $u_{\theta\theta} = 0$.

Solutions to Exercises 6.3

1. Using Proposition 6.3.1 the solution will be

$$\begin{aligned} u(re^{i\theta}) &= 1 - r\cos(\theta) + r^2\sin(2\theta) \\ &= 1 - r\cos(\theta) + 2r^2\sin(\theta)\cos(\theta) \\ &= 1 - x + 2xy \end{aligned}$$

5. We must solve $1 - x + 2xy = T$ which gives us the result of $y = \frac{T+x-1}{2x}$.

Exercises 6.4 For exercises 1-4, we will use the fact that the Fourier series is written as

$$f(\theta) = a_0 + a_1\cos(\theta) + b_1\sin(\theta) + a_2\cos(2\theta) + b_2\sin(2\theta) + a_3\cos(3\theta) + b_3\sin(3\theta) + \cdots$$

One can compute these coefficients as integrals to confirm the result.

1. Here we have that

$$\begin{aligned} f(\theta) &= 1 - \cos(\theta) + \sin(2\theta) \\ &= 1 + (-1)\cos(\theta) + (0)\sin(\theta) + (0)\cos(2\theta) + (1)\sin(2\theta) + \cdots \end{aligned}$$

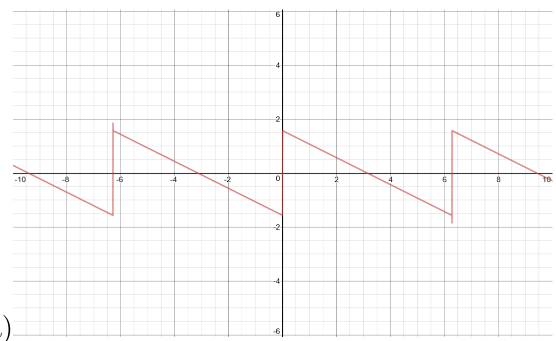
This gives us that $a_0 = 1$, $a_1 = -1$, and $b_2 = 1$, where the rest of the coefficients are 0.

5. Using the integral definition of the Fourier coefficients we will find that

$$\begin{aligned} a_0 &= \pi \\ a_n &= 0 \\ b_n &= \frac{-2}{n} \end{aligned}$$

where we use integration by parts and the facts that

$$\cos(n\theta) = \Big(\frac{\sin(n\theta)}{n}\Big)' \qquad \text{and} \qquad \sin(n\theta) = \Big(-\frac{\cos(n\theta)}{n}\Big)'.$$

9. (a) 

(b) Deriving the Fourier coefficients we get

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - \theta)d\theta \\ &= \frac{1}{4\pi}\left[\pi\theta - \frac{1}{2}\theta^2\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_0^{2\pi} (\pi\cos(n\theta) - \theta\cos(n\theta))d\theta \\ &= \frac{1}{2\pi}\left[\frac{1}{n^2}(n(\pi - \theta)\sin(n\theta) - \cos(n\theta))\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_0^{2\pi} (\pi - \theta)\sin(n\theta)d\theta \\ &= \frac{1}{2\pi}\left[-\frac{1}{n^2}(\sin(n\theta) + n(\pi - \theta)\cos(n\theta))\right]_0^{2\pi} \\ &= \frac{1}{n} \end{aligned}$$

This implies that $f(\theta) = \sum_1^\infty \frac{\sin(n\theta)}{n}$.

13. Using the integral definition of the Fourier coefficients we will find that

$$a_0 = \frac{2}{\pi}$$

$$a_n = -\frac{4\cos\left(\frac{n\pi}{2}\right)}{\pi(n^2 - 1)}$$

$$b_n = 0$$

which gives the desired result.

Solutions to Exercises 7.1

1. An analytic function $f(z)$ is conformal where $f'(z) \neq 0$. If $f(z) = \frac{z^2+1}{e^z}$, then

$$f'(z) = e^{-z}(-z^2 + 2z - 1).$$

We have

$$f'(z) = 0 \Rightarrow z^2 - 2z + 1 = 0 \Rightarrow z = 1.$$

Thus f is conformal at all $z \neq 1$.

5. The function $f(z) = z + \frac{1}{z}$ is analytic at all $z \neq 0$, and then $f'(z) = 1 - \frac{1}{z^2}$. So

$$f'(z) = 0 \Rightarrow 1 - \frac{1}{z^2} = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1.$$

Thus $f(z)$ is conformal at all $z \neq 0, \pm 1$.

9. The function $\sin z$ is entire. Its derivative $\cos z$ is nonzero except for $z = \frac{\pi}{2} + k\pi$. Hence $f(z) = \sin z$ is conformal at all $z \neq \frac{\pi}{2} + k\pi$. In particular, $f(z)$ is conformal at $z = 0$, $\pi + i\,a$, $i\pi$. At these points, $f(z)$ rotates by an angle $\arg \cos z$ and scales by a factor of $|\cos z|$. For $z = 0$, we have $\cos 0 = 1$. Thus f rotates by an angle $\arg 1 = 0$ and scales by a factor of 1. Thus at $z = 0$, $\sin z$ acts like the identity map $z \mapsto z$.

For $z = \pi + i\,a$, we have $\cos(\pi + i\,a) = -\cos(ia) = -\cosh a$. Thus f rotates by an angle $\arg(-\cosh a) = \pi$ and scales by a factor of $\cosh a$.

For $z = i\pi$, we have $\cos i\pi = \cosh \pi$. Thus f rotates by an angle $\arg(\cosh \pi) = 0$ and scales by a factor of $\cosh \pi$.

13. The lines $x = a$ and $y = b$ are clearly orthogonal and they intersect at the point $a + ib$. Their images by a mapping $f(z)$ are two curves that intersect at the point $f(a + ib)$. The image curves will be orthogonal at the point $f(a + ib)$ if $f(z)$ is conformal at $a + ib$. Hence it is enough to check that $f(z)$ is analytic and $f'(z) \neq 0$ at $z = a + ib$, in order to conclude that the image curves are orthogonal at $f(a + ib)$. In the case of $f(z) = e^z$, the image curves will always be orthogonal, because $f'(z) \neq 0$ for all z. Indeed, the image of the line $x = a$ (or $z = a + iy$, $-\infty < y < \infty$) is the circle $w = e^{a+iy} = e^a e^{iy}$, with center at 0 and radius e^a. The image of the line $y = b$ (or $z = x + ib, -\infty < x < \infty$) is the ray at angle b, $w = e^x e^{ib}$. The ray and the circle intersect at right angle at the point e^{a+ib}.

17. (a) We have

$$J\Big(\frac{1}{z}\Big) = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{\frac{1}{z}}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) = J(z).$$

(b) Fix $R > 1$ and let $S_R = \{z : |z| = R,\ 0 \leq \operatorname{Arg} z\}$. Write the image of a point $z = Re^{i\theta}$ on S_R as $w = u + iv$. Then

$$\begin{aligned} w &= J(Re^{i\theta}) = \frac{1}{2}\left(Re^{i\theta} + \frac{1}{R}e^{-i\theta}\right) \\ &= \frac{1}{2}\left(R(\cos\theta + i\sin\theta) + +\frac{1}{R}(\cos\theta - i\sin\theta)\right) \\ &= \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\theta + i\frac{1}{2}\left(R - \frac{1}{R}\right)\sin\theta. \end{aligned}$$

So

$$u = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\theta \quad \text{and} \quad v = \frac{1}{2}\left(R - \frac{1}{R}\right)\sin\theta;$$

$$\frac{u}{\frac{1}{2}\left(R + \frac{1}{R}\right)} = \cos\theta \quad \text{and} \quad \frac{v}{\frac{1}{2}\left(R - \frac{1}{R}\right)} = \sin\theta.$$

As θ varies from 0 to π, w traces the upper part of the ellipse

$$\frac{u^2}{\left[\frac{1}{2}\left(R+\frac{1}{R}\right)\right]^2}+\frac{v^2}{\left[\frac{1}{2}\left(R-\frac{1}{R}\right)\right]^2}=\cos^2\theta+\sin^2\theta=1.$$

Solutions to Exercises 7.2

1. Consider the linear fractional transformation (LFT)

$$\phi(z) = i\frac{1-z}{1+z}.$$

(a) We have

$$\phi(1) = 0, \quad \phi(0) = i, \quad \phi(i) = i\frac{1-i}{1+i} = 1.$$

(b) Let L_1 denote the line through $z_1 = 1$ and $z_2 = 0$, and L_2 the line through $z_2 = 0$ and $z_3 = i$. We know that the image of a line by a LFT is either a line or a circle. So to determine whether $\phi[L_j]$ is a line or a circle, it suffices to check whether the images of three points on L_j are colinear. Another way to determine the image of a line by a LFT is to check whether the image is bounded or unbounded. If it is unbounded, then it is necessarily a line. For example, the point $z = -1$ is on L_1. Since $\phi(-1) = \infty$, we conclude that $\phi[L_1]$ is a line through the points $\phi(z_1) = 0$ and $\phi(z_2) = i$; that is, $\phi[L_1]$ is the imaginary axis in the w-plane.

Since L_2 is perpendicular to L_1 and ϕ is conformal at $z = 0$, the point of intersection, $\phi[L_1]$ and $\phi[L_2]$ must be orthogonal at $\phi[0]$, their point of intersection. Now $\phi[L_2]$ goes through the points i and 1. In order for $\phi[L_2]$ to be perpendicular to the imaginary axis, $\phi[L_1]$, it must be a circle that goes through the points 1 and i. To get a third point on this circle, take $z = -i$ on L_2. Then

$$\phi(-i) = i\frac{1+i}{1-i} = -1.$$

5. The inverse of the LFT

$$w = \phi(z) = i\frac{1-z}{1+z} = \frac{i-iz}{1+z} = \frac{iz+i}{z+1}$$

is given by (2), where $a = b = i$, $c = d = 1$:

$$z = \psi(w) = \frac{z-i}{-z+i}.$$

Let us check the images of w_j:

$$\begin{aligned}\psi(w_1) &= \psi(0) = 1 = z_1;\\ \psi(w_2) &= \psi(i) = 0 = z_2;\\ \psi(w_3) &= \psi(1) = i = z_3.\end{aligned}$$

9. Suppose $c \neq 0$ and consider the equation

$$\phi(z) = z \quad \text{or} \quad \frac{az+b}{cz+d} = z \qquad \left(z \neq -\frac{d}{c}\right).$$

This equation is equivalent to

$$cz^2 + (d-a)z - b = 0,$$

which has at most two solutions. Thus in this case we have at most two distinct fixed points. If $c = 0$, the LFT takes the form $\phi(z) = \alpha z + \beta$, where $\alpha \neq 0$. This linear function is either the identity ($\alpha = 1$) or has one fixed point, which is obtained by solving $z = \alpha z + \beta$, $z = \frac{\beta}{1-\alpha}$. In all cases, the LFT has at most two fixed points.

13. The circle in the first figure is centered at $1 + i$ and has radius 1. To map it to the circle shown in the second figure, use the translation

$$f_1(z) = z - 1 - i.$$

To map the inside of the unit circle to its outside, and its outside to its inside, as shown in the mapping from the second to the third figure, use an inversion

$$f_2(w_1) = \frac{1}{w_1}.$$

Note that if $|z| = 1$, then $|f_2(z)| = |1/z| = 1$, so f_2 maps the unit circle to the unit circle. Since it takes boundary to boundary, and 0 to ∞, it will act as we claimed. Finally, to map the unit disk to the upper half-plane, use the LFT

$$f_3(w_2) = i\frac{1 - w_2}{1 + w_2}$$

(see Example 7.2.3(a)). The desired mapping is

$$\begin{aligned} w &= f_3 \circ f_2 \circ f_1(z) = i\frac{1 - w_2}{1 + w_2} = i\frac{1 - \frac{1}{w_1}}{1 + \frac{1}{w_1}} \\ &= i\frac{w_1 - 1}{w_1 + 1} = i\frac{z - 1 - i - 1}{z - 1 - i + 1} = i\frac{z - 2 - i}{z - i}. \end{aligned}$$

17. (a) We have

$$f_1(z) = \sin(z), \quad f_2(w_1) = \frac{i - w_1}{i + w_1}.$$

(b) For the boundary, consider the point $z = 0$ and notice that $f_1(0) = 0$ and $f_2(0) = 1$. For the interior, consider the point $z = i$ and notice that

$$f_1(i) = \sin(i) = \frac{1 - e^2}{2ei} = \Big(\frac{e^2 - 1}{2e}\Big)i,$$

which is in the interior of the region in the w_1 plane as $\mathrm{Im}(f_1(i)) \geq 0$. Notice

$$f_2\Big(\frac{e^2 - 1}{2ei}\Big) = \frac{2e - e^2 + 1}{2e + e^2 - 1},$$

which is on the interior of the unit disk.
(c) The composition is given by

$$g(z) = f_2(f_1(z)) = \frac{i - \sin(z)}{i + \sin(z)}.$$

21. Consider the LFT

$$f_1(z) = i\frac{1 - z}{1 + z}.$$

We know from Example 1(a) that f_1 takes the unit disk onto the upper half-plane. What does it do to the upper semi-disk? The lower boundary of the semi-disk, the interval $[-1, 1]$ is perpendicular to the upper semi-circle at the point 1. Since f_1 is conformal at $z = 1$, the images of the interval $[-1, 1]$ and semi circle intersect at right angle. Since they both go through the point -1, which is mapped to ∞, we conclude that these images are perpendicular lines. Using $f_1(0) = i$ and $f_1(i) = 1$, we conclude that $[-1, 1]$ is mapped to the upper half of the imaginary axis, and the semi circle is mapped to the right half of the real axis. Finally, testing the image of an interior point, say $i/2$, we find that $f_1(\frac{i}{2}) = \frac{4}{5} + \frac{3}{5}i$, which is a point in the first quadrant. With this we conclude that the upper semi-disk is mapped to the first quadrant, since boundary is mapped to boundary and interior points to interior points.

To go from the first quadrant to a horizontal strip, we use a logarithmic mapping, because the logarithm maps the punctured plane onto a fundamental strip of width 2π (see Sec. 1.7.) For our purpose, it is easy to check that $\operatorname{Log} z$ will work: The semi-line $(0, \infty)$ is mapped to the line $(-\infty, \infty)$ and the semi-line, $z = iy$, $0 < y < \infty$, is mapped to $\operatorname{Log}(iy) = \ln y + i\frac{\pi}{2}$, which describes the horizontal line shown in the figure. The composition mapping is

$$w = \operatorname{Log}\left(i\frac{1-z}{1+z}\right).$$

25. (a) Take $-1 < \alpha < 1$ and let $\phi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$. Since α is real, we have $\phi_\alpha(z) = \frac{z-\alpha}{1-\alpha z}$. We know from Proposition 4.6.2 that ϕ_α maps the unit disk onto itself and takes the unit circle onto itself. As it is explained in Example 7.2.9, in order to center C_1, it is enough to choose α so that $\phi_\alpha(a) = -\phi_\alpha(b)$. Equivalently,

$$\begin{aligned} \frac{a-\alpha}{1-\alpha a} &= -\frac{b-\alpha}{1-\alpha b}; \\ (a-\alpha)(1-\alpha b) &= -(b-\alpha)(1-\alpha a); \\ (a+b)\alpha^2 - 2(1+ab)\alpha + a + b = 0 &\Rightarrow \alpha^2 - 2\frac{1+ab}{a+b}\alpha + 1 = 0. \end{aligned}$$

Note that $a + b \neq 0$. Solving this quadratic equation in α, we obtain with roots

$$\alpha_1 = \frac{1+ab}{a+b} + \sqrt{\left(\frac{1+ab}{a+b}\right)^2 - 1} \quad \text{and} \quad \alpha_2 = \frac{1+ab}{a+b} - \sqrt{\left(\frac{1+ab}{a+b}\right)^2 - 1}.$$

We next proceed to show that $\alpha_1 > 1$ and $0 < \alpha_2 < 1$. So we must choose α_2 in constructing ϕ_α.
(b) We have $|a| < 1$, $0 < b < 1$ and $0 \le |a| < b < 1$. Since $1 - b > 0$, if $0 \le a < 1$, then multiplying $1 - b$ by a, we get $1 - b > a(1 - b)$. If $-1 < a < 0$, then $1 - b > a(1 - b)$, because the right side is negative, while the left side is positive. In all cases, $1 + ab > a + b > 0$. Consequently,

$$\left(\frac{1+ab}{a+b}\right)^2 > 1;$$

hence the discriminant of the quadratic equation in α in part (a) is positive and so we have two distinct roots, α_1 and α_2.
(c) From the inequality $1 + ab > a + b > 0$, we conclude that $\frac{1+ab}{a+b} > 1$ and so $\alpha_1 > 1$. The product of $\alpha_1 \cdot \alpha_2 = 1$, as can be checked directly or by using a well-known property that the product of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $\frac{c}{a}$. Since $1 < \alpha_1$, we conclude that $0 < \alpha_2 < 1$, in order to have the equality $\alpha_1 \cdot \alpha_2 = 1$.
(d) Now arguing exactly as we did in Example 4, we conclude that $\phi(z) = \frac{z-\alpha}{1-\alpha z}$ with

$$\alpha = \frac{1+ab}{a+b} - \sqrt{\left(\frac{1+ab}{a+b}\right)^2 - 1}$$

will map C_2 onto C_2, C_1 onto a circle centered at the origin with radius $r = \phi(b)$, and the region between C_2 and C_1 onto the annular region bounded by $\phi[C_1]$ and the unit circle.

29. From the figure, the first mapping is a translation:

$$f_1(z) = z - \frac{3}{2}i.$$

The second mapping is a linear fractional transformation that maps the shaded half plane to the disk of radius 2 and center at the origin. This mapping is the inversion

$$f_2(w_1) = \frac{1}{w_1}.$$

To see this, recall that the mapping is conformal except at 0. So it will map perpendicular lines and circles to the same. The image of a circle centered at the origin, with radius R, is another circle centered at the origin with radius $1/R$. Thus the circle in the second figure is mapped onto the circle with center at the origin and radius 2. The region inside the circle is mapped outside the region of the image circle and vice versa. In particular, the shaded half-plane is mapped inside the circle with radius 2. We must show that the image of the lower half-plane, which is also mapped inside the circle of radius 2, is the the smaller disk shown in the 3rd figure. To this end, we check the image of the horizontal boundary of this lower half-plane, which is the line through $-\frac{3}{2}i$. This line is mapped to a circle (the image of the line is contained in a bounded region, so it has to be a circle and not a line). Moreover, this circle makes an angle 0 with the real axis (the real axis is mapped to the real axis by an inversion). Since $f_2(\infty) = 0$ and $f_2(-\frac{3}{2}i) = \frac{2}{3}i$, we conclude that the image of the lower clear half-plane is the clear smaller disk as shown in the 3rd figure.

To construct the last mapping of the sequence, we will appeal to the result of Exercise 25. Let us prepare the ground for the application of this result, by rotating the picture in the 3rd figure by $-\frac{\pi}{2}$, and then scaling by $\frac{1}{2}$. This amounts to multiplying by $\frac{1}{2}e^{-\frac{\pi}{2}}$ or $\frac{-i}{2}$. So we introduce the mapping $f_3(w_2) = \frac{1}{2}e^{-\frac{\pi}{2}}w_2$. This will map the outer circle in the 3rd figure to the unit circle and the inner circle to a circle of radius $\frac{1}{6}$ and center at $\frac{1}{6}$. In the notation of Exercise 25, we have $a = 0$ and $b = \frac{1}{3}$. According to Exercise 25, the mapping that will center the inner circle is

$$\phi_\alpha(w_3) = \frac{w_3 - \alpha}{1 - \alpha w_3},$$

where α is the smaller of the roots of

$$\alpha^2 - 2\frac{1}{\frac{1}{3}}\alpha + 1 = 0 \Rightarrow \alpha^2 - 6\alpha + 1 = 0.$$

Thus $\alpha = 3 - 2\sqrt{2}$. Composing all the mappings together, we obtain:

$$\begin{aligned}
f(z) &= \frac{w_3 - \alpha}{1 - \alpha w_3} = \frac{\frac{-i}{2}w_2 - \alpha}{1 - \alpha\frac{-i}{2}w_2} \\
&= \frac{-iw_2 - 2\alpha}{2 + i\alpha w_2} = \frac{\frac{-i}{w_1} - 2\alpha}{2 + i\frac{\alpha}{w_1}} \\
&= \frac{-i - 2\alpha w_1}{2w_1 + i\alpha} \\
&= \frac{-i - 2\alpha(z - \frac{3}{2}i)}{2(z - \frac{3}{2}i) + i\alpha} = \frac{-2\alpha z - i(1 - 3\alpha)}{2z + i(-3 + \alpha)} \\
&= \frac{-2(3 - 2\sqrt{2})z - i(1 - 3(3 - 2\sqrt{2}))}{2z - i2\sqrt{2}} \\
&= \frac{-(3 - 2\sqrt{2})z + i(4 - 3\sqrt{2})}{z - i\sqrt{2}}.
\end{aligned}$$

33. (a) If $w = u + iv$ and $z = x + iy$, then

$$u + iv = w = \frac{1}{z} = \frac{1}{x + iy}\frac{x - iy}{x - iy} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

Thus, $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = \frac{-y}{x^2+y^2}$.
(b) When $z = \frac{1}{w}$, we obtain that

$$x + iy = \frac{1}{u + iv}\frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} + i\frac{-v}{u^2 + v^2}.$$

Thus,

$$x(u,v) = \frac{u}{u^2+v^2} \qquad \text{and} \qquad y(u,v) = \frac{-v}{u^2+v^2}.$$

37. Let S be a line that passes through the origin. We know that S is of the form

$$A(x^2+y^2)+Bx+Cy+D=0$$

where $B^2+C^2-4AD>0$ where $A=0$ as S is a line. Moreover, S passes through the origin and so $D=0$ by part (d) of Exercise 34. By Exercise 35, we get that $f[S]$ is of the form

$$D(u^2+v^2)+Bu-Cv+A=0.$$

But, $D=0$ and $A=0$ and so $f[S]$ is also a line that passes through the origin. (b) The image of $f(z_0)$ of any nonzero point $z_0 \in S$ uniquely determines the line of $f[S]$ because we know $f[S]$ passes through the origin, which will allow us to compute the slope of $f[S]$ and thus the equation of $f[S]$.

Solutions to Exercises 7.3

1. Transform the problem to the upper half-plane as in Example 7.3.1 using the linear fractional transformation

$$\phi(z) = i\frac{1-z}{1+z}.$$

The LFT ϕ takes the unit disk onto the upper-half plane, the upper semi-circle onto the positive real axis, and the lower semi-circle onto the negative real axis. Thus the problem in the upper half of the w-plane becomes:

$$\Delta U = 0, \quad U(\alpha) = 70 \text{ if } \alpha > 0,\ U(\alpha) = 50 \text{ if } \alpha < 0.$$

The solution in the w-plane is $U(w) = a \operatorname{Arg}(w) + b$, where a and b must be chosen in according to the boundary conditions: $a \operatorname{Arg}(w) + b = 70$, when $\operatorname{Arg} = 0$ and $a \operatorname{Arg}(w) + b = 50$, when $\operatorname{Arg} = \pi$. Thus $b = 70$ and $a = -\frac{20}{\pi}$. Hence $U(w) = -\frac{20}{\pi} \operatorname{Arg} w + 70$ and so

$$u(z) = U(\phi(z)) = -\frac{20}{\pi} \operatorname{Arg}\left(i\frac{1-z}{1+z}\right) + 70.$$

5. The mapping $f_1(z) = z^4$ takes the shaded region in Figure 7.56 onto the upper semi-disk of radius 1. The upper semi-disk is mapped onto the first quadrant by

$$f_2(w_1) = i\frac{1-w_1}{1+w_1}.$$

Thus the mapping

$$w = \phi(z) = i\frac{1-z^4}{1+z^4}$$

takes the shaded region in Figure 7.56 onto the first quadrant. It is also a conformal mapping, being the composition of such mappings. We now determine the image of the boundary. The rays at angle 0 and $\pi/4$ are mapped onto the interval $[-1, 1]$ by f_1. The quarter of a circle is mapped to the upper semi-circle by f_1. The mapping f_2 takes the interval $[-1, 1]$ onto the upper half of the imaginary axis, and the semi-circle onto the right half of the real axis.

Thus the problem in the first quadrant of the w-plane becomes:

$$\Delta U = 0, \quad U(\alpha) = 100 \text{ if } \alpha > 0,\ U(i\alpha) = 0 \text{ if } \alpha > 0.$$

The solution in the w-plane is $U(w) = a \operatorname{Arg}(w) + b$, where a and b must be chosen in according to the boundary conditions: $a \operatorname{Arg}(w) + b = 100$, when $\operatorname{Arg} = 0$ and $a \operatorname{Arg}(w) + b = 0$, when $\operatorname{Arg} = \frac{\pi}{2}$. Thus $b = 100$ and $a = -\frac{200}{\pi}$. Hence $U(w) = -\frac{200}{\pi} \operatorname{Arg} w + 100$ and so

$$u(z) = U(\phi(z)) = -\frac{200}{\pi} \operatorname{Arg}\left(i\frac{1-z^4}{1+z^4}\right) + 100.$$

9. To solve the given problem, we can proceed as in Example 7.3.8 and make the necessary changes. A much quicker way is based is to use the solution in Example 7.3.8 and superposition, as follows. Let $u_1(z)$ denote the solution in Example 7.3.8. Let $u_2(z) = 100$. It is clear that u_2 is harmonic for all z. Thus $u_2 - u_1$ is harmonic in the shaded region of Fig. 18. On the real axis, we have $u_2(z) - u_1(z) = 100 - 0 = 100$. On the upper semi-circle, we have $u_2(z) - u_1(z) = 100 - 100 = 0$. Thus $100 - u_1$ is the solution, where u_1 is the solution in Example 7.3.8.

13. This problem is very similar to that in Example 7.3.3. The first step is to map the region to an annular region bounded by concentric circles. This can be done by using the linear fractional transformation

$$\phi(z) = \frac{4z-1}{4-z}.$$

Then

$$\phi(z) = \frac{4z-1}{4-z} = \frac{z-\frac{1}{4}}{1-\frac{z}{4}} = \phi_{\frac{1}{4}}(z).$$

Using ϕ, we map the outer circle to the unit circle and the inner circle to the circle of radius $\frac{1}{4}$ and center at 0. As in Example 7.3.3, the solution is

$$U(w) = 100 + 100\frac{\ln|w| + \ln 4}{\ln(1/4)} = 100 - 100\frac{\ln|w| + \ln 4}{\ln 4} = -100\frac{\ln|w|}{\ln 4}.$$

Thus the solution is

$$u(z) = -\frac{100}{\ln 4}\ln\left|\frac{4z-1}{4-z}\right|.$$

17. (a) The solution of the Dirichlet Problem in Figure 7.6.7 is obtained by applying the Poisson formula (7.3.5), with $f(s) = s$ if $-1 < s < 1$, $f(s) = -1$ if $s < -1$ and $f(s) = 1$ if $s > 1$:

$$\begin{aligned} u(x+iy) &= \frac{y}{\pi}\int_{-1}^{1}\frac{s}{(x-s)^2+y^2}ds + \frac{y}{\pi}\int_{-\infty}^{-1}\frac{-1}{(x-s)^2+y^2}ds + \frac{y}{\pi}\int_{1}^{\infty}\frac{ds}{(x-s)^2+y^2} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We compute each integral separately:

$$\begin{aligned} I_1 &= -\frac{y}{\pi}\int_{-1}^{1}\frac{x-s}{(x-s)^2+y^2}ds + \frac{y}{\pi}\int_{-1}^{1}\frac{x}{(x-s)^2+y^2}ds \\ &= \frac{y}{\pi}\frac{1}{2}\ln\left((x-s)^2+y^2\right)\Big|_{-1}^{1} + \frac{xy}{\pi}\int_{-1}^{1}\frac{ds}{(s-x)^2+y^2} \\ &= -\frac{y}{2\pi}\left[\ln\left((x+1)^2+y^2\right) - \ln\left((x-1)^2+y^2\right)\right] + \frac{x}{y\pi}\int_{-1}^{1}\frac{ds}{\left(\frac{s-x}{y}\right)^2+1} \\ &= -\frac{y}{2\pi}\ln\frac{(x+1)^2+y^2}{(x-1)^2+y^2} + \frac{x}{\pi}\int_{s=-1}^{1}\frac{du}{u^2+1} \qquad (\text{Let } u = \frac{s-x}{y}.) \\ &= -\frac{y}{2\pi}\ln\frac{(x+1)^2+y^2}{(x-1)^2+y^2} + \frac{x}{\pi}\tan^{-1}\left(\frac{s-x}{y}\right)\Big|_{s=-1}^{1} \\ &= -\frac{y}{2\pi}\ln\frac{(x+1)^2+y^2}{(x-1)^2+y^2} + \frac{x}{\pi}\left[\tan^{-1}\left(\frac{1+x}{y}\right) + \tan^{-1}\left(\frac{1-x}{y}\right)\right]. \end{aligned}$$

Similarly, we find

$$\begin{aligned} I_2 &= \frac{y}{\pi}\int_{-\infty}^{-1}\frac{-1}{(x-s)^2+y^2}ds \\ &= -\frac{1}{2} + \frac{y}{\pi}\tan^{-1}\left(\frac{1+x}{y}\right); \\ I_3 &= \frac{y}{\pi}\int_{1}^{\infty}\frac{ds}{(x-s)^2+y^2} \\ &= \frac{1}{2} - \frac{y}{\pi}\tan^{-1}\left(\frac{1-x}{y}\right). \end{aligned}$$

Thus

$$u(x+iy) = -\frac{y}{2\pi}\ln\frac{(x+1)^2+y^2}{(x-1)^2+y^2} + \frac{1+x}{\pi}\tan^{-1}\left(\frac{1+x}{y}\right) + \frac{-1+x}{\pi}\tan^{-1}\left(\frac{1-x}{y}\right).$$

(b) To solve the problem in Figure 7.68, we reduce to the problem in Figure 7.67 by using the mapping $\sin z$. This yields the solution

$$\begin{aligned}
u(x+iy) \quad &= \quad -\frac{\operatorname{Im}(\sin z)}{\pi} \ln \frac{(\operatorname{Re}(\sin z)+1)^2 + (\operatorname{Im}(\sin z))^2}{(\operatorname{Re}(\sin z)-1)^2 + (\operatorname{Im}(\sin z))^2} \\
&\quad +\frac{1+\operatorname{Re}(\sin z)}{\pi} \tan^{-1}\left(\frac{1+\operatorname{Re}(\sin z)}{\operatorname{Im}(\sin z)}\right) \\
&\quad +\frac{-1+\operatorname{Re}(\sin z)}{\pi} \tan^{-1}\left(\frac{1-\operatorname{Re}(\sin z)}{\operatorname{Im}(\sin z)}\right) \\
&= \quad -\frac{\cos x \sinh y}{\pi} \ln \frac{(\sin x \cosh y+1)^2 + (\cos x \sinh y)^2}{\sin x \cosh y - 1)^2 + (\cos x \sinh y)^2} \\
&\quad +\frac{1+\sin x \cosh y}{\pi} \tan^{-1}\left(\frac{1+\sin x \cosh y}{\cos x \sinh y}\right) \\
&\quad +\frac{-1+\sin x \cosh y}{\pi} \tan^{-1}\left(\frac{1-\sin x \cosh y}{\cos x \sinh y}\right),
\end{aligned}$$

where we have used the fact that $\operatorname{Re} \sin z = \sin x \cosh y$ and $\operatorname{Im} \sin z = \cos x \sinh y$.

21. First we will show that

$$1 - |\psi(z)|^2 = \frac{4y}{x^2 + (1+y)^2}.$$

Recall that $\psi(z) = \frac{i-z}{i+z}$. Finding the modulus we get

$$|\psi(z)| = \left|\frac{i-z}{i+z}\right| = \left|\frac{i-x-iy}{i+x+iy}\right| = \left|\frac{-x+i(1-y)}{x+i(1+y)}\right| = \sqrt{\frac{x^2+(1-y)^2}{x^2+(1+y)^2}}\Bigg|.$$

Then we can conclude

$$\begin{aligned}
1 - |\psi(z)|^2 &= 1 - \frac{x^2+(1-y)^2}{x^2+(1+y)^2} \\
&= \frac{x^2+(1+y)^2 - x^2 - (1-y)^2}{x^2+(1+y)^2} \\
&= \frac{1+2y+y^2-1+2y-y^2}{x^2+(1+y)^2} \\
&= \frac{4y}{x^2+(1+y)^2}
\end{aligned}$$

as desired. Next we will show that

$$|e^{i\phi} - \psi(z)|^2 = |\psi(s) - \psi(z)|^2 = 4\frac{(x-s)^2+y^2}{(1+s^2)(x^2+(1+y)^2)}.$$

Calculating we see

$$\begin{aligned}
|e^{i\phi} - \psi(z)|^2 &= |\psi(s) - \psi(z)|^2 \\
&= \left|\frac{i-s}{i+s} - \frac{i-z}{i+z}\right|^2 \\
&= \left|\frac{(i-s)(i+z) - (i-z)(i+s)}{(i+s)(i+z)}\right|^2 \\
&= \left|\frac{-2i(s-z)}{(i+s)(i+z)}\right|^2 \\
&= \left|\frac{-2i(s-x-iy)}{sz+is+iz-1}\right|^2 \\
&= \left|\frac{-2is+2ix-2y}{sx+isy+is+ix-y-1}\right|^2 \\
&= \left|\frac{-2y+i(2x-2s)}{sx-y-1+i(s+sy+x)}\right|^2 \\
&= \frac{(-2y)^2+(2x-2s)^2}{(sx-y-1)^2+(s+sy+x)^2} \\
&= \frac{4((x-s)^2+y^2)}{(1+s)^2(x^2+y^2+2y+1)} \\
&= 4\frac{(x-s)^2+y^2}{(1+s)^2(x^2+(1+y)^2)}
\end{aligned}$$

which, once combined with the result derived above will give us the desired result.

25. To solve the Dirichlet problem in the upper half-plane with boundary function

$$f(x) = \frac{1}{x^4+1},$$

we appeal to the Poisson formula, which gives: for $y > 0$ and $-\infty < x < \infty$,

$$u(x,\, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+s^4}\,\frac{1}{(s-x)^2+y^2}\,ds.$$

We evaluate the integral by using the residue techniques of Section 5.3. Let

$$f(z) = \frac{1}{1+z^4}\,\frac{1}{(z-x)^2+y^2}.$$

By Proposition 5.3.4 we have

$$u(x,\, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+s^4}\,\frac{1}{(s-x)^2+y^2}\,ds = 2\pi i\frac{y}{\pi}\sum_j \operatorname{Res}(f, z_j) = 2iy\sum_j \operatorname{Res}(f, z_j),$$

where the sum extends over the residues of f in the upper half-plane. We have

$$z^4 = -1 \Rightarrow z^4 = e^{i\pi} \Rightarrow z_1 = e^{i\frac{\pi}{4}}, z_2 = e^{i\frac{3\pi}{4}}, z_3 = e^{i\frac{5\pi}{4}}, z_4 = e^{i\frac{7\pi}{4}}.$$

Only z_1 and z_2 are in the upper half-plane. At these points, f has simple poles and the residues there are computed with the help of Proposition 5.1.3: We have

$$\operatorname{Res}(f, z_1) = \frac{1}{4z^3}\Big|_{z=z_1}\,\frac{1}{(z-x)^2+y^2}\Big|_{z=z_1} = \frac{1}{4e^{i\frac{3\pi}{4}}}\,\frac{1}{(e^{i\frac{\pi}{4}}-x)^2+y^2} = R_1;$$

$$\operatorname{Res}(f, z_2) = \frac{1}{4z^3}\Big|_{z=z_2} \frac{1}{(z-x)^2+y^2}\Big|_{z=z_2} = \frac{1}{4e^{i\frac{9\pi}{4}}} \frac{1}{(e^{i\frac{3\pi}{4}}-x)^2+y^2} = \frac{1}{4e^{i\frac{\pi}{4}}} \frac{1}{(e^{i\frac{3\pi}{4}}-x)^2+y^2} = R_2.$$

The equation $(z-x)^2+y^2=0$ has two roots $z = x \pm iy$, as you can check. Only $x+iy$ is in the upper half-plane, since $y>0$. The residue of f at $x+iy$ is:

$$\operatorname{Res}(f, x+iy) = \frac{1}{z^4+1}\Big|_{z=x+iy} \frac{1}{2(z-x)}\Big|_{z=x+iy} = \frac{1}{(x+iy)^4+1} \frac{1}{2iy} = R_3.$$

Thus,

$$u(x,\, y) = 2iy\Big(\frac{1}{4e^{i\frac{3\pi}{4}}} \frac{1}{(e^{i\frac{\pi}{4}}-x)^2+y^2} + \frac{1}{4e^{i\frac{\pi}{4}}} \frac{1}{(e^{i\frac{3\pi}{4}}-x)^2+y^2} + \frac{1}{(x+iy)^4+1} \frac{1}{2iy}\Big).$$

This answer should be real-valued, because the Poisson integral involves real-valued functions only. This is not obvious, but the integral can be simplified using Mathematica to yield an expression that is obviously real-valued.

29. We appeal to the Poisson formula which gives: for $y>0$ and $-\infty < x < \infty$,

$$u(x,\, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \sin a(x-s)\, \frac{1}{s^2+y^2}\, ds.$$

This integral follows the approach in Example 5.4.1

$$\sin\big(a(x-s)\big)\, \frac{1}{s^2+y^2} = \frac{\sin(ax)\cos(as)}{s^2+y^2} - \frac{\cos(ax)\sin(as)}{s^2+y^2}.$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} \sin\big(a(x-s)\big)\, \frac{1}{s^2+y^2}\, ds &= \int_{-\infty}^{\infty} \frac{\sin(ax)\cos(as)}{s^2+y^2} ds - \int_{-\infty}^{\infty} \frac{\cos(ax)\sin(as)}{s^2+y^2} ds \\
&= \sin(ax) \int_{-\infty}^{\infty} \frac{\cos(as)}{s^2+y^2} ds - \cos(ax) \overbrace{\int_{-\infty}^{\infty} \frac{\sin(as)}{s^2+y^2} ds}^{=0} \\
&= \sin(ax) \int_{-\infty}^{\infty} \frac{\cos(as)}{s^2+y^2} ds,
\end{aligned}$$

where one of the integrals is 0 because the integrand is an odd function. We now appeal to Example 5.4.1 (doing the cases $a>0$ and $a<0$ separately):

$$u(x,\, y) = \frac{y}{\pi}\sin(ax)\int_{-\infty}^{\infty} \frac{\cos(as)}{s^2+y^2} ds = \frac{y\sin(ax)}{\pi}\frac{\pi}{y} e^{-|a|y} = e^{-|a|y}\sin(ax).$$

33. The conformal mapping that takes the first quadrant to the upper half-plane is $w = \phi(z) = z^2$. It also maps the interval $[0,\, 1]$ in the z-plane to the interval $[0,\, 1]$ in the w-plane. To determine the boundary values in the w-plane, we compose the inverse of ϕ with the boundary values in the z-plane. This gives the value 0 for all points on the real axis outside the interval $[0,\, 1]$, and for the w in the interval $[0,\, 1]$, the boundary value is $(\sqrt{w})^2 = w$. We thus obtain the boundary values in the w-plane: $f(w) = w$ if $0 < w < 1$ and 0 otherwise. The solution in the w-plane is obtained by applying the Poisson integral formula. We will use coordinates $(\alpha,\, \beta)$ in the w-plane. Thus

$$\begin{aligned}
U(\alpha,\, \beta) &= \frac{\beta}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(\alpha-s)^2+\beta^2}\, ds = \frac{\beta}{\pi}\int_0^1 \frac{s}{(\alpha-s)^2+\beta^2}\, ds \\
&= \frac{\beta}{\pi}\Big[\frac{\alpha}{\beta}\tan^{-1}\Big(\frac{s-\alpha}{\beta}\Big) + \frac{1}{2}\ln\big(\beta^2+(s-\alpha)^2\big)\Big]_0^1 \\
&= \frac{\alpha}{\pi}\Big[\tan^{-1}\Big(\frac{1-\alpha}{\beta}\Big) + \tan^{-1}\Big(\frac{\alpha}{\beta}\Big)\Big] + \frac{\beta}{2\pi}\ln\frac{\beta^2+(1-\alpha)^2}{\beta^2+\alpha^2},
\end{aligned}$$

where the integral in s was evaluated by similar methods as in Exercise 17. The solution in the z-plane is obtained by replacing α by $\operatorname{Re}\phi(z) = \operatorname{Re} z^2 = x^2 - y^2$ and β by $\operatorname{Im}\phi(z) = \operatorname{Im} z^2 = 2xy$. Thus $u(x, y)$ equals

$$\frac{x^2 - y^2}{\pi}\left[\tan^{-1}\left(\frac{1-(x^2-y^2)}{2xy}\right) + \tan^{-1}\left(\frac{x^2-y^2}{2xy}\right)\right] + \frac{xy}{\pi}\ln\frac{(2xy)^2 + \big(1-(x^2-y^2)\big)^2}{(2xy)^2 + (x^2-y^2)^2}.$$

Exercises 7.4

1. The outside angles at w_1, w_2, and w_3 are $\theta_1 = \frac{\pi}{2}$, $\theta_2 = -\pi$, and $\theta_3 = \frac{\pi}{2}$, respectively. By (7.4.7), we have

$$(z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}} = i(1-z^2)^{\frac{1}{2}}.$$

Applying (7.4.2), we have

$$\begin{aligned} f(z) &= A\int (z-1)^{-\frac{1}{2}}(z+1)^{-\frac{1}{2}} z\,dz + B \\ &= A\int i(1-z^2)^{-\frac{1}{2}} z\,dz + B \\ &= -Ai(1-z^2)^{\frac{1}{2}} + B \end{aligned}$$

The fact that $f(\pm 1) = 0$ implies that $B = 0$. The fact that $f(0) = 1$ and $B = 0$ implies that $A = i$. Therefore, we conclude $f(z) = \sqrt{1-z^2}$.

5. The outside angles at w_1 and w_2 are $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = -\frac{\pi}{2}$, respectively. By (7.4.7), we have $(z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}} = i(1-z^2)^{\frac{1}{2}}$. Applying (7.4.2), we find

$$\begin{aligned} f(z) &= A\int (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}\,dz + B \\ &= iA\int (1-z^2)^{\frac{1}{2}}\,dz + B \\ &= \frac{iA}{2}\left[z\sqrt{1-z^2} + \sin^{-1} z\right] + B, \end{aligned}$$

where in evaluating the last integral we used integration by parts. Using $f(1) = -1$, we obtain

$$\frac{Ai\pi}{4} + B = -1.$$

Using $f(-1) = 1$, we obtain

$$\frac{-Ai\pi}{4} + B = 1.$$

Thus, $B = 0$ and $A = \frac{4i}{\pi}$. Therefore, we conclude

$$f(z) = -\frac{2}{\pi}\left[z\sqrt{1-z^2}\sin^{-1} z\right].$$

9. (a) Suppose $\theta_n < \pi$ and $\theta_1 + \theta_2 + \cdots + \theta_n = 2\pi$. It follows that

$$2\pi = \theta_1 + \theta_2 + \cdots + \theta_n < \theta_1 + \theta_2 + \cdots + \theta_{n-1} + \pi,$$

so we conclude that $\pi < \theta_1 + \theta_2 + \cdots + \theta_{n-1}$ by subtracting π from both sides of the inequality above. We conclude that $\theta_1 + \theta_2 + \cdots + \theta_{n-1} > \pi$. We let $\beta_j = \frac{\theta_j}{\pi}$.

(b) Because A and B in (7.4.1) just dilate, rotate, and translate the mapping, A and B do not affect the convergence of $f(z)$ or the closure of the polygon. Hence, we can choose $A = 1$ and $B = 0$. Let $x_0 = 1 + \max_{1 \le j \le n-1} x_j$ and set

$$f(z) = \int_{[x_0,z]} \frac{d\zeta}{(\zeta - x_1)^{\beta_1}(\zeta - x_2)^{\beta_2} \cdots (\zeta - x_{n-1})^{\beta_{n-1}}}.$$

(c) Let $x_0 = 1 + \max_{1 \le j \le n-1} x_j$ and $z = x$ be real. We will use the limit comparison test for the integrand to show that

$$\lim_{y \to \infty} f(y) = \int_{x_0}^{\infty} \frac{dx}{(x - x_1)^{\beta_1}(x - x_2)^{\beta_2} \cdots (x - x_{n-1})^{\beta_{n-1}}}$$

is finite. Let $a = \max_{1 \le j \le n-1} x_j$ and $b = \min_{1 \le j \le n-1} x_j$. Observe that

$$\begin{aligned} x - a + a - b &= (x - a)\left(1 + \frac{a - b}{x - a}\right) \\ &\le (x - a)(1 + a - b) \end{aligned}$$

because $x - a \ge 1$. This observation gives us

$$(1 + a - b)(x - a) \ge x - a + a - b = x - b \ge x - x_j \ge x - a$$

for all $1 \le j \le n - 1$. We see that $(x - x_j)^{\beta_j} \approx (x - a)^{\beta_j}$ for all β_j. Using this, we have

$$\int_{x_0}^{\infty} \frac{dx}{(x - x_1)^{\beta_1}(x - x_2)^{\beta_2} \cdots (x - x_{n-1})^{\beta_{n-1}}} \le C \int_{x_0}^{\infty} \frac{dx}{(x - a)^{\beta_1 + \beta_2 + \cdots + \beta_{n-1}}},$$

where C is a constant. Specifically, we have

$$C = \prod_{j : \beta_j > 0} (1 + a - b)^{\beta_j}.$$

Notice that

$$\beta_1 + \beta_2 + \cdots + \beta_{n-1} = \frac{\theta_1 + \theta_2 + \cdots + \theta_{n-1}}{\pi} > \frac{\pi}{\pi} = 1$$

from part (a). Therefore, after a change in variables, we see that the integral on the right converges. (d) Let

$$w_n = \lim_{y \to \infty} f(y) = \int_{x_0}^{\infty} \frac{dx}{(x - x_1)^{\beta_1}(x - x_2)^{\beta_2} \cdots (x - x_{n-1})^{\beta_{n-1}}}$$

from above. We want to show that $\lim_{|z| \to \infty} f(z) = w_n$ for any $z \in \mathbb{C}$. We will prove this for $z = Re^{i\theta}$ where $R > 0$ and $0 \le \theta \le \pi$. A similar argument will hold for $z = Re^{i\theta}$ where

$R > 0$ and $\pi \le \theta \le 2\pi$. First, observe that

$$\begin{aligned}
|f(Re^{i\theta}) - f(R)| &= \Bigg| \int_{[x_0, Re^{i\theta}]} \frac{dx}{(x-x_1)^{\beta_1}(x-x_2)^{\beta_2}\cdots(x-x_{n-1})^{\beta_{n-1}}} \\
&\qquad - \int_{[x_0,R]} \frac{dx}{(x-x_1)^{\beta_1}(x-x_2)^{\beta_2}\cdots(x-x_{n-1})^{\beta_{n-1}}} \Bigg| \\
&= \Bigg| \int_{[x_0, Re^{i\theta}]} \frac{dx}{(x-x_1)^{\beta_1}(x-x_2)^{\beta_2}\cdots(x-x_{n-1})^{\beta_{n-1}}} \\
&\qquad + \int_{[R,x_0]} \frac{dx}{(x-x_1)^{\beta_1}(x-x_2)^{\beta_2}\cdots(x-x_{n-1})^{\beta_{n-1}}} \Bigg| \\
&= \Bigg| \int_{[R, Re^{i\theta}]} \frac{dx}{(x-x_1)^{\beta_1}(x-x_2)^{\beta_2}\cdots(x-x_{n-1})^{\beta_{n-1}}} \Bigg| \\
&\le R\pi M(R),
\end{aligned}$$

where $M(R)$ is the maximum of the absolute value of the integrand on the upper semicircle of radius R. Also, the length of the segment joining R to $Re^{i\theta}$ is largest when it is the circumference of the upper semicircle or $R\pi$. Next, we claim that $\lim_{R\to\infty} M(R)R^{\beta_1+\cdots+\beta_{n-1}} = 1$. Notice that for $R > \max_{1\le j\le n-1} |x_j|$ we have

$$\begin{aligned}
M(R) &= \sup_{|z|=R} \left| \frac{1}{(z-x_1)^{\beta_1}\cdots(z-x_{n-1})^{\beta_{n-1}}} \right| \\
&= \sup_{|z|=R} \left[\prod_{\substack{1\le j\le n-1 \\ \beta_j \ge 0}} \frac{1}{|z-x_j|^{\beta_j}} \prod_{\substack{1\le j\le n-1 \\ \beta_j < 0}} \frac{1}{|z-x_j|^{\beta_j}} \right] \\
&= \sup_{|z|=R} \left[\prod_{\substack{1\le j\le n-1 \\ \beta_j \ge 0}} \frac{1}{|z-x_j|^{|\beta_j|}} \prod_{\substack{1\le j\le n-1 \\ \beta_j < 0}} |z-x_j|^{|\beta_j|} \right] \\
&= \prod_{\substack{1\le j\le n-1 \\ \beta_j \ge 0}} \frac{1}{(R-x_j)^{|\beta_j|}} \prod_{\substack{1\le j\le n-1 \\ \beta_j < 0}} (R+x_j)^{|\beta_j|},
\end{aligned}$$

since the first product becomes maximum for $z = R$ and the second product becomes maximum for $z = -R$ and both of these numbers lie on the circle $|z| = R$. This implies that

$$\lim_{R\to\infty} M(R)R^{\beta_1+\cdots+\beta_{n-1}} = \lim_{R\to\infty} \prod_{\substack{1\le j\le n-1 \\ \beta_j \ge 0}} \frac{R^{|\beta_j|}}{(R-x_j)^{|\beta_j|}} \prod_{\substack{1\le j\le n-1 \\ \beta_j < 0}} \frac{(R+x_j)^{|\beta_j|}}{R^{|\beta_j|}}.$$

But we have $\beta_i \ge 0$

$$\lim_{R\to\infty} \frac{R^{\beta_i}}{(R-x_i)^{\beta_i}} = 1$$

and for $\beta_j < 0$,

$$\lim_{R\to\infty} \frac{(x_j+R)^{|\beta_j|}}{R^{|\beta_j|}} = 1.$$

We conclude that $\lim_{R\to\infty} M(R) = 1$. Note that

$$RM(R) = R^{1-(\beta_1+\cdots+\beta_{n-1})}(M(R)R^{\beta_1+\cdots+\beta_{n-1}}).$$

The fact that $\beta_1 + \cdots + \beta_{n-1} > 1$ by our discussion in part (c) implies that $RM(R) \to 0$ as $R \to \infty$. This implies that $|f(Re^{i\theta}) - f(R)| \to 0$ uniformly in θ as $R \to \infty$. The fact that $f(R) \to w_n$ as $R \to \infty$ implies that $f(z) \to w_n$ as $|z| \to \infty$.

Solutions to Exercises 7.5

1. We provide two ways of solving this exercise. First, we may derive the Green's function for Ω directly. The transformation

$$\phi(z) = \frac{z-1}{z+1}$$

is a one-to-one analytic function taking Ω onto the unit disk. For fixed $z \in \Omega$, the transformation

$$\Phi(z,\zeta) = \frac{\phi(\zeta)-\phi(z)}{1-\overline{\phi(z)}\phi(\zeta)} = -\frac{(\overline{z}+1)(z-\zeta)}{(z+1)(\overline{z}+\zeta)}$$

is also a one-to-one analytic function taking Ω onto the unit disk with $\Phi(z,z) = 0$. Hence, the Green's function for Ω is

$$\begin{aligned} G(z,\zeta) &= \ln|\Phi(z,\zeta)| \\ &= \ln\frac{|z-\zeta|}{|\overline{z}+\zeta|} \\ &= \frac{1}{2}\ln\frac{|z-\zeta|^2}{|\overline{z}+\zeta|^2} \\ &= \frac{1}{2}\ln\frac{(x-s)^2+(y-t)^2}{(x+s)^2+(y-t)^2}, \end{aligned}$$

where $z = x+iy$ and $\zeta = s+it$ are in Ω.

Alternatively, we know from Example 7.5.5 that the Green's function for the upper half-plane $\tilde{\Omega}$ is

$$\tilde{G}(z,\zeta) = \ln\frac{|z-\zeta|}{|\overline{z}-\zeta|} = \frac{1}{2}\ln\frac{(x-s)^2+(y-t)^2}{(x-s)^2+(y+t)^2},$$

where $z = x+iy$ and $\zeta = s+it$ are in $\tilde{\Omega}$. The rotation map

$$\rho(z) = iz$$

is a one-to-one analytic function taking Ω onto $\tilde{\Omega}$. Hence, the Green's function for Ω is

$$\begin{aligned} G(z,\zeta) &= \tilde{G}(\rho(z),\rho(\zeta)) \\ &= \ln\frac{|iz-i\zeta|}{|\overline{iz}-i\zeta|} \\ &= \ln\frac{|z-\zeta|}{|\overline{z}+\zeta|} \\ &= \frac{1}{2}\ln\frac{(x-s)^2+(y-t)^2}{(x+s)^2+(y-t)^2}, \end{aligned}$$

where $z = x+iy$ and $\zeta = s+it$ are in Ω.

5. We know from Example 7.5.5 that the Green's function for the upper half-plane $\tilde{\Omega}$ is

$$\tilde{G}(z,\zeta) = \ln\frac{|z-\zeta|}{|\overline{z}-\zeta|} = \frac{1}{2}\ln\frac{(x-s)^2+(y-t)^2}{(x-s)^2+(y+t)^2},$$

where $z = x+iy$ and $\zeta = s+it$ are in $\tilde{\Omega}$. Observe that the map

$$\phi(z) = e^z$$

is a one-to-one analytic function taking Ω onto $\tilde{\Omega}$. Hence, the Green's function for Ω is

$$\begin{aligned} G(z,\zeta) &= \tilde{G}(\phi(z),\phi(\zeta)) \\ &= \ln \frac{\left|e^z - e^\zeta\right|}{\left|e^{\overline{z}} - e^\zeta\right|} \\ &= \frac{1}{2}\ln\frac{(e^x\cos y - e^s\cos t)^2 + (e^x\sin y - e^s\sin t)^2}{(e^x\cos y - e^s\cos t)^2 + (e^x\sin y + e^s\sin t)^2}, \end{aligned}$$

where $z = x + iy$ and $\zeta = s + it$ are in Ω.

9. (a) Green's function in the first quadrant, Ω, is derived in Exercise 3: For z and ζ in Ω,

$$G(z,\,\zeta) = \ln\frac{|z-\zeta|\,|z+\zeta|}{|\overline{z}-\zeta|\,|\overline{z}+\zeta|}.$$

Write $z = x + iy$, $\zeta = s + it$, where x, y, s, and t are positive. Then

$$\begin{aligned} G(z,\,\zeta) &= \frac{1}{2}\ln\frac{\big((x-s)^2+(y-t)^2\big)\big((x+s)^2+(y+t)^2\big)}{\big((x-s)^2+(-y-t)^2\big)\big((x+s)^2+(-y+t)^2\big)} \\ &= \frac{1}{2}\ln\big((x-s)^2+(y-t)^2\big) + \frac{1}{2}\ln\big((x+s)^2+(y+t)^2\big) \\ &\quad -\frac{1}{2}\ln\big((x-s)^2+(y+t)^2\big) - \frac{1}{2}\ln\big((x+s)^2+(-y+t)^2\big). \end{aligned}$$

The boundary of Ω consists of two half-lines: the positive real axis, and the upper part of the imaginary axis. On the imaginary axis, the normal derivative is minus the derivative with respect to s. Thus, on the imaginary axis,

$$\begin{aligned} \frac{\partial}{\partial n}G(z,\,\zeta) &= -\frac{\partial}{\partial s}G(z,\,\zeta)\Big|_{s=0} \\ &= -\frac{1}{2}\frac{\partial}{\partial s}\ln\big((x-s)^2+(y-t)^2\big) - \frac{1}{2}\frac{\partial}{\partial s}\ln\big((x+s)^2+(y+t)^2\big) \\ &\quad +\frac{1}{2}\frac{\partial}{\partial s}\ln\big((x-s)^2+(y+t)^2\big) + \frac{1}{2}\frac{\partial}{\partial s}\ln\big((x+s)^2+(-y+t)^2\big)\Big|_{s=0} \\ &= \frac{(x-s)}{(x-s)^2+(y-t)^2} - \frac{(x+s)}{(x+s)^2+(y+t)^2} \\ &\quad -\frac{(x-s)}{(x-s)^2+(y+t)^2} + \frac{(x+s)}{(x+s)^2+(-y+t)^2}\Big|_{s=0} \\ &= \frac{x}{x^2+(y-t)^2} - \frac{x}{x^2+(y+t)^2} - \frac{x}{x^2+(y+t)^2} + \frac{x}{x^2+(-y+t)^2} \\ &= \frac{2x}{x^2+(y-t)^2} - \frac{2x}{x^2+(y+t)^2}. \end{aligned}$$

By a similar argument, we find that, on the real axis,

$$\begin{aligned} \frac{\partial}{\partial n}G(z,\,\zeta) &= -\frac{\partial}{\partial t}G(z,\,\zeta)\Big|_{t=0} \\ &= \frac{2y}{(x-s)^2+y^2} - \frac{2y}{(x+s)^2+y^2}. \end{aligned}$$

From (7.5.9) we have

$$u(z) = \frac{1}{2\pi}\int_\Gamma u(\zeta)\frac{\partial}{\partial n}G(z,\,\zeta)ds.$$

We have to figure our the symbols in this integral. For the part of the boundary that is on the imaginary axis, $ds = dt$, and $u(\zeta) = g(t)$. For the part of the boundary that is on the real axis, $u(\zeta) = f(s)$, and $ds = ds$ (bad notation: on the left, ds stands for element of arc length; on the right ds stands for element of integration with respect to the s variables.) Using the values of the normal derivative, we get

$$\begin{aligned} u(z) &= \frac{y}{\pi}\int_0^{\infty} f(s)\left(\frac{1}{(x-s)^2+y^2} - \frac{1}{(x+s)^2+y^2}\right) ds \\ &\quad + \frac{x}{\pi}\int_0^{\infty} g(t)\left(\frac{1}{x^2+(y-t)^2} - \frac{1}{x^2+(y+t)^2}\right) dt. \end{aligned}$$

(b) Consider the special case in which $g(t) = 0$. Call the solution in this u_1. From part (a), we have

$$u_1(z) = \frac{y}{\pi}\int_0^{\infty} f(s)\left(\frac{1}{(x-s)^2+y^2} - \frac{1}{(x+s)^2+y^2}\right) ds, \quad z = x+iy.$$

We will show how to derive this solution by reducing the problem to a Dirichlet problem in the upper half-plane. Indeed, consider the Dirichlet problem in the upper half-plane with boundary values $u(x, 0) = f(x)$ if $x > 0$ and $u(x, 0) = -f(-x)$ if $x < 0$. Thus the boundary function is the odd extension of $f(x)$ to the entire real line. We will use the same notation for the odd extension as for the function f. Then, by the Poisson integral formula on the real line,

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\, ds, \quad y > 0,\ -\infty < x < \infty.$$

To determine the values of u_1 on the upper part of the imaginary axis, we set $x = 0$ and get

$$u(0, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{s^2+y^2}\, ds = 0,$$

because f is odd. So u is harmonic in the upper half-plane; is equal to 0 on the imaginary axis; and is equal to $f(x)$ on the x-axis. Therefore, its restriction to the first quadrant solves the Dirichlet problem with boundary values 0 on the imaginary axis and $f(x)$ on the positive real axis. Thus u agrees with u_1 in the first quadrant. Because f is odd, we can write

$$\begin{aligned} u(x, y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\, ds \\ &= \frac{y}{\pi}\int_{-\infty}^{0} \frac{f(s)}{(x-s)^2+y^2}\, ds + \frac{y}{\pi}\int_0^{\infty} \frac{f(s)}{(x-s)^2+y^2}\, ds \\ &= -\frac{y}{\pi}\int_{\infty}^{0} \frac{f(-s)}{(x+s)^2+y^2}\, ds + \frac{y}{\pi}\int_0^{\infty} \frac{f(s)}{(x-s)^2+y^2}\, ds \\ &= \frac{y}{\pi}\int_0^{\infty} \frac{f(s)}{(x+s)^2+y^2}\, ds + \frac{y}{\pi}\int_0^{\infty} \frac{f(s)}{(x-s)^2+y^2}\, ds \\ &= \frac{y}{\pi}\int_0^{\infty} f(s)\Big(\frac{1}{(x+s)^2+y^2} + \frac{1}{(x-s)^2+y^2}\Big) ds, \end{aligned}$$

which is precisely the formula for u_1 that we obtained earlier by using Green's function in the first quadrant.

(c) The case in Figure 7.115 in which $f(s) = 0$ is very similar to the case treated in part (b). We just have to interchange x and y, s and t. The solution in this case is

$$u_2(x, y) = \frac{x}{\pi}\int_0^{\infty} g(t)\Big(\frac{1}{(y+t)^2+x^2} + \frac{1}{(y-t)^2+x^2}\Big) dt,$$

which is again the formula that you obtain by setting $f = 0$ in part (a).
(d) The Dirichlet problem in Figure 7.115 can be written as the "sum" of two Dirichlet problems: the problem in which $g = 0$, and the problem in which $f = 0$. The first problem is solved in (9b) (the solution is u_1), and the second problem is solved in (9c) (the solution is u_2). It is easy to straightforward to verify that $u = u_1 + u_2$ is the solution of the problem in Figure 7.115. Adding the two formulas that we have for u_1 and u_2, we obtain the formula for u that we derived in (9a).

The point of this problem is that we were able to solve the Dirichlet problem in the first quadrant by reducing the problem to two problems in the upper half-plane and then using the Poisson integral formula on the real line.

Solutions to Exercises 7.6

1. Applying Proposition 7.6.12 with

$$\phi(z) = i(z-1),$$

which is a one-to-one analytic function taking Ω onto the upper half-plane, the Nuemann function for Ω is

$$\begin{aligned}
N(z,\zeta) &= \ln|\phi(z)-\phi(\zeta)| + \ln|\overline{\phi(z)}-\phi(\zeta)| \\
&= \ln|i(z-1)-i(\zeta-1)| + \ln|-i(\overline{z}-1)-i(\zeta-1)| \\
&= \ln|z-\zeta| + \ln|\overline{z}+\zeta-2| \\
&= \frac{1}{2}\ln\left((x-s)^2+(y-t)^2\right) + \frac{1}{2}\ln\left((x+s-2)^2+(-y+t)^2\right),
\end{aligned}$$

where $z = x+iy$ and $\zeta = s+it$ are in Ω.

5. Applying Proposition 7.6.12 with

$$\phi(z) = \sin z,$$

which is a one-to-one analytic function taking Ω onto the upper half-plane, the Nuemann function for Ω is

$$\begin{aligned}
N(z,\zeta) &= \ln|\phi(z)-\phi(\zeta)| + \ln|\overline{\phi(z)}-\phi(\zeta)| \\
&= \ln|\sin z - \sin\zeta| + \ln|\overline{\sin z} - \sin\zeta| \\
&= \frac{1}{2}\ln\left((\sin x\cosh y - \sin s\cosh t)^2 + (\cos x\sinh y - \cos s\sinh t)^2\right) \\
&\quad + \frac{1}{2}\ln\left((\sin x\cosh y - \sin s\cosh t)^2 + (\cos x\sinh y + \cos s\sinh t)^2\right),
\end{aligned}$$

where $z = x+iy$ and $\zeta = s+it$ are in Ω.

9. Recall that Ω is a region bounded by simple path Γ and $G(z,\zeta)$ is the Green's function for Ω. Fix $z\in\Omega$. For $\epsilon>0$, let $\Omega_\epsilon = \Omega\setminus\overline{B_\epsilon(z)}$. Write

$$G(z,\zeta) = u_1(z,\zeta) + \ln|z-\zeta|$$

for $\zeta\neq z$ in Ω, where $u_1(z,\zeta)$ is harmonic on Ω with $u_1(z,\zeta) = -\ln|z-\zeta|$ on Γ. Moreover, $\Delta u = h$ on Ω. So $u_1(z,\zeta)$ and $h(\zeta)$ are bounded in $B_\epsilon(z)$. Now, we justify (7.6.8) as follows:

$$\begin{aligned}
\Big|\iint_{\Omega_\epsilon} G(z,\zeta)h(\zeta)\,dA - \iint_{\Omega} G(z,\zeta)h(\zeta)\,dA\Big| &= \Big|\iint_{B_\epsilon(z)} G(z,\zeta)h(\zeta)\,dA\Big| \\
&\leq \iint_{B_\epsilon(z)} |u_1(z,\zeta)h(\zeta)|\,dA + \iint_{B_\epsilon(z)} |\ln|z-\zeta|h(\zeta)|\,dA \\
&\leq C_1\iint_{B_\epsilon(z)} dA + C_2\iint_{B_\epsilon(z)} \ln|z-\zeta|\,dA \\
&= C_1\epsilon^2\pi + C_2\int_0^{2\pi}\int_0^{\epsilon} r\ln r\,dr d\theta \\
&= C_1\epsilon^2\pi + C_2\int_0^{2\pi}\lim_{\delta\to0^+}\left[\frac{r^2\ln r}{2} - \frac{r^2}{4}\right]_{r=\delta}^{r=\epsilon} d\theta \\
&= C_1\epsilon^2\pi + C_2 2\pi\left(\frac{\epsilon^2\ln\epsilon}{2} - \frac{\epsilon^2}{4}\right),
\end{aligned}$$

which converges to 0 as $\epsilon \to 0^+$. This shows that

$$\iint_{\Omega_\epsilon} G(z,\zeta)h(\zeta)\,dA \longrightarrow \iint_{\Omega} G(z,\zeta)h(\zeta)\,dA \quad \text{as } \epsilon \to 0^+.$$

13. Let Ω be a region bounded by simple path Γ, and let $N(z,\zeta)$ be a Neumann function for Ω. So for fixed $z \in \Omega$, the outward unit directional derivative of $N(z,\zeta)$ on Γ is a constant $C = \frac{2\pi}{L}$, where $L = \int_\Gamma ds$.

13. (a) For a constant A, select

$$\begin{aligned} F(z) &:= \frac{1}{L}\int_\Gamma N(z,\zeta)\,ds - \frac{A}{L} \\ &= \frac{C}{2\pi}\int_\Gamma N(z,\zeta)\,ds - \frac{AC}{2\pi}, \end{aligned}$$

for $z \in \Omega$. Then, by part (c) of Exercise 12,

$$N_0(z,\zeta) = N(z,\zeta) - F(z)$$

is also a Neumann function for Ω. Moreover,

$$\begin{aligned} \int_\Gamma N_0(z,\zeta)\,ds &= \int_\Gamma N(z,\zeta)\,ds - \int_\Gamma F(z)\,ds \\ &= \int_\Gamma N(z,\zeta)\,ds - F(z)L \\ &= A. \end{aligned}$$

That is, the integral of $N_0(z,\zeta)$ over the boundary Γ is a constant.
(b) Let $z_1, z_2 \in \Omega$. For $\epsilon > 0$, denote $\Omega_\epsilon = \Omega \setminus \left(\overline{B_\epsilon(z_1)} \cup \overline{B_\epsilon(z_2)}\right)$, and its boundary is denoted by Γ_ϵ Applying Green's sencond identity to $N(z_1,\zeta)$ and $N(z_2,\zeta)$ over Ω_ϵ, we have

$$\begin{aligned} 0 &= \int_{\Gamma_\epsilon} N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds - \int_{\Gamma_\epsilon} N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds \\ &= \int_\Gamma N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds + \int_{\partial B_\epsilon(z_1)} N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds + \int_{\partial B_\epsilon(z_2)} N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds \\ &\quad - \int_\Gamma N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds + \int_{\partial B_\epsilon(z_1)} N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds + \int_{\partial B_\epsilon(z_2)} N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds \end{aligned}$$

Using the same argument as in Exercise 10, we can show that as $\epsilon \to 0^+$

$$\begin{aligned} \int_{\partial B_\epsilon(z_1)} N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds &\longrightarrow 0, \\ \int_{\partial B_\epsilon(z_2)} N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds &\longrightarrow 0, \\ \int_{\partial B_\epsilon(z_2)} N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds &\longrightarrow -2\pi N(z_1,z_2), \\ \int_{\partial B_\epsilon(z_1)} N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds &\longrightarrow -2\pi N(z_2,z_1). \end{aligned}$$

Letting $\epsilon \to 0^+$, we arrive at

$$0 = \int_\Gamma N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds - 2\pi N(z_1,z_2) \quad - \int_\Gamma N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds + 2\pi N(z_2,z_1).$$

That is

$$\begin{aligned} 2\pi N(z_1,z_2) - 2\pi N(z_2,z_1) &= \int_\Gamma N(z_1,\zeta)\frac{\partial N(z_2,\zeta)}{\partial n}\,ds \quad - \int_\Gamma N(z_2,\zeta)\frac{\partial N(z_1,\zeta)}{\partial n}\,ds \\ &= C\int_\Gamma N(z_1,\zeta)\,ds \quad - C\int_\Gamma N(z_2,\zeta)\,ds. \end{aligned}$$

Hence,

$$\begin{aligned} N_0(z_1,z_2) &= N(z_1,z_2) - F(z_1) \\ &= N(z_1,z_2) - \frac{C}{2\pi}\int_\Gamma N(z_1,\zeta)\,ds - \frac{AC}{2\pi} \\ &= N(z_2,z_1) - \frac{C}{2\pi}\int_\Gamma N(z_2,\zeta)\,ds - \frac{AC}{2\pi} \\ &= N(z_2,z_1) - F(z_2) \\ &= N_0(z_2,z_1). \end{aligned}$$

This proves that the Nuemann function $N_0(z,\zeta)$ for Ω is symmetric.
(c) Replacing $N(z,\zeta)$ by $N_0(z,\zeta)$ in (7.6.23), we have

$$\begin{aligned} \frac{1}{2\pi}\iint_\Omega h(\zeta)N_0(z,\zeta)\,dA - \frac{1}{2\pi}\int_\Gamma f(\zeta)N_0(z,\zeta)\,ds &= \left[\frac{1}{2\pi}\iint_\Omega h(\zeta)N(z,\zeta)\,dA - \frac{1}{2\pi}\int_\Gamma f(\zeta)N_(z,\zeta)\,ds\right] \\ &\quad + \left[\frac{F(z)}{2\pi}\iint_\Omega h(\zeta)\,dA - \frac{F(z)}{2\pi}\int_\Gamma f(\zeta)\,ds\right] \\ &= u(z), \end{aligned}$$

where the last equality holds by (7.6.22) and (7.6.23).

http://www.springer.com/978-3-319-94062-5

Complex Analysis with Applications
Asmar, N.H.; Grafakos, L.
2018, VIII, 494 p. 389 illus., 4 illus. in color., Hardcover
ISBN: 978-3-319-94062-5